On the well-posedness of a dispersal model for farmers and hunter-gatherers in the Neolithic transition

JÁN ELIAŠ*[1], KABIR HUMAYUN†[2], AND MASAYASU MIMURA‡[2,3]

[1]*Institute of Mathematics and Scientific Computing, University of Graz, Heinrichstrasse 36, 8010 Graz, Austria*
[2]*Meiji Institute for Advanced Study of Mathematical Sciences, Meiji University, 4-21-1 Nakano, Nakano ku, Tokyo, Japan, 164–8525*
[3]*Department of Mathematical Engineering, Musashino University, 3-3-3 Ariake, Koto-ku, Tokyo, Japan, 135–8181*

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Abstract

We study the reaction-diffusion model that consists of equations that govern the spatio-temporal evolution of sedentary and migrating farmers and hunter-gatherers in the Neolithic transition. Ecologically, the model stems from the fact that a lifestyle of agriculture and settlement, as it allows for a larger population, is evolutionary advantageous than hunting and gathering. Therefore, in our modelling framework, we assume that farmers do not migrate unless the population density pressure forces them. We prove the global well-posedness of the system and, in contrast to the previous modelling work on the transition from hunting and gathering to farming, we show numerically that for a suitable value of a "stay-or-migrate" threshold the model is capable of reproducing the rate of spread of farming that corresponds to the archeological findings in Europe.

Keywords: Mathematical modelling; Neolithic transition; reaction-diffusion system; well-posedness.

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1 Introduction

Mathematical modelling plays an increasingly important role in ecology and evolution. Good mathematical models with advanced data analysis improve our understanding of how the dynamics of species, including human populations are determined by fundamental biological conditions and processes. The Neolithic transition in Europe involving the demographic shift from hunting and gathering to farming is one of such examples that can be fruitfully modelled mathematically.

*Email: jan.elias@uni-graz.at
†Email: hkabirju@gmail.com
‡Email: mimura.masayasu@gmail.com

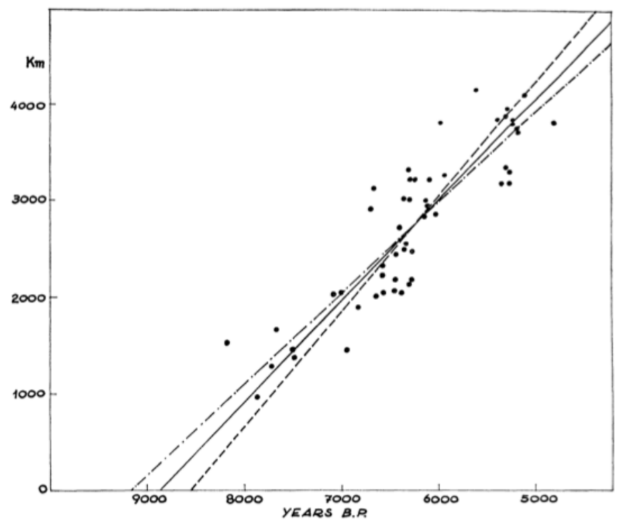

Figure 1: Spread of early farming in Europe. The site of Jericho is taken as the presumed centre of diffusion in this case. The points correspond to sites providing estimates of the time of arrival of early farming in different parts of Europe [2, 3]. The figure taken from [2], reprinted by permission from the Royal Anthropological Institute of Great Britain and Irelands.

The Neolithic transition in Europe started around 10000 years ago. Radiocarbon dating has produced a large amount of evidence regarding this transition in Europe. In particular, it follows from the data that the transition from foraging to agriculture was steadily shifted with an almost constant velocity that is roughly estimated to be 0.8∼1.2 km/year, see Fig. 1 [2, 3, 17]. The question that arises is why and how the Neolithic transition evolved with constant velocity? In order to answer this question theoretically, as a natural extension of the Fisher-KPP equation [13], the following reaction-diffusion model involving the conversion of hunter-gatherers to farmers was proposed in [2, 3]:

$$\begin{cases} F_t = d_F \Delta F + r_F(1 - F/K_F)F + e_F FH, \\ H_t = d_H \Delta H + r_H(1 - H/K_H)H - e_H FH, \end{cases} \tag{1}$$

where $F(x,t)$ and $H(x,t)$ are, respectively, the densities of the farmer and hunter-gatherer populations at the time $t > 0$ and the position x in the space domain; d_F, r_F and K_F (resp., d_H, r_H and K_H) are, respectively, the random dispersal rate, the intrinsic growth rate and the carrying capacity of the farming (resp., hunting-gathering) population. The acculturation (or conversion) rates between hunter-gatherers and farmers are denoted by e_F and e_H. All these rates are positive constants.

Let us consider (1) in a bounded domain Ω. We assume that F and G satisfy homogeneous Neumann boundary conditions as well as the initial conditions

$$F(\mathbf{x},0) = F_0(\mathbf{x}), \quad H(\mathbf{x},0) = H_0(\mathbf{x}), \tag{2}$$

where F_0 and G_0 are nonnegative and F_0 is compactly supported in Ω. It can be shown both numerically and analytically (see, e.g., [19]) that, whenever $e_H K_F > r_H$, then the spatially constant equilibrium $(0, K_H)$ is unstable, whereas $(K_F, 0)$ is asymptotically stable. Therefore (1) is a monostable system. Ecologically, this implies that hunter-gatherers are completely converted into farmers, which is in close agreement with the observations [3]. Thus, the system (1) seems to be a plausible model for the transition from hunting-gathering into farming. A question of interest is then whether the observed transition velocity of 0.8∼1.2 km/year can be predicted by this model? For this question, we have the following analytical result: the one dimensional system (1)-(2) with the boundary conditions

$$(F,H)(-\infty) = (K_F, 0), \quad (F,H)(+\infty) = (0, K_H),$$

which emerge from the observation that the initial population of hunter-gatherers is completely replaced by the population of farmers after large time, possesses indeed a travelling wave solution with the minimal velocity c^* given by

$$c^* = 2\sqrt{d_F(r_F + e_F K_H)},$$

see [4]. It is shown in [2] that $d_F = 61.76$ km^2/year and $r_F = 0.032$/year, respectively. Whence the speed of the farming spread predicted by the model (1) is larger than 2.8 km/year and, consequently, much faster that the observed transition velocity of about 1 km/year on average.

For this problem, Fort studied the dispersal of farmers from the available data and gave the estimate $d_F = 15.44$ km^2/year [16, 14]. However, this implies that $c^* > 2\sqrt{d_F r_F} = 1.4$ km/year which is still larger than the observation. Then Fort and Méndez proposed a time-delayed Fisher-KPP equation which takes into account the newborn children of farmers that usually can migrate only after some time, namely, after they are grown-up [15]. They showed that the velocity of the traveling wave solution of this equation is given by

$$c = \frac{2\sqrt{d_F r_F}}{1 + \tau r_F/2}$$

where τ is the residence time, i.e., the time interval between the migration of parents and, presumably, the subsequent migration of children [15]. As a result, the velocity of the farming spread can be slowed down by taking τ suitably large (say, $\tau = 25$ years).

In this paper we aim to propose a new model for the Neolithic transition from hunter-gatherers to farmers which will possibly possess expanding velocity that is close to the observed one. Our motivation for the 'farmer-hunter'-modelling is that farmers when settled in a favourable environment for farming prefer to stay there unless they are forced to move. The driving force for migration, which we have in mind, is the farmer overcrowding. Therefore, our essential assumptions on the migration of farmers and hunter-gatherers are as follows:

(A1) Farmers have a sedentary lifestyle;

(A2) If the density of the sedentary farmers becomes high, some of them start migrating and dispersing randomly, as if the population pressure effect occurs;

(A3) Sedentary and migrating farmers convert to each other depending on the total density of the farmers;

(A4) Hunter-gatherers always disperse randomly, independently of the farmers.

We remark that (A1)-(A3) are the essential assumptions which are different from the assumptions in the previous models. In addition to (A1)-(A4), the dynamics of farmers and hunter-gatherers is assumed to obey the logistic growth and the conversion of hunter-gatherers into farmers is taken to be proportional to the population densities.

In view of (A1)-(A4), we propose the following macroscopic three-component nondimensionalized system:

$$
\begin{cases}
F_{1t} = (1-F)F_1 + s_1 F_1 H - \dfrac{1}{\varepsilon}\left(p(F)F_1 - (1-p(F))F_2\right), \\
F_{2t} = d\Delta F_2 + (1-F)F_2 + s_2 F_2 H + \dfrac{1}{\varepsilon}\left(p(F)F_1 - (1-p(F))F_2\right), \\
H_t = \Delta H + b(1-H)H - g_1 F_1 H - g_2 F_2 H,
\end{cases}
\tag{3}
$$

where F_1, F_2 and H represent the densities of sedentary farmers, migrating farmers and hunter-gatherers, respectively, and $F = F_1 + F_2$ is the total density of farmers. Further, b is the intrinsic growth rate for the population of hunter-gatherers, s_1 and g_1 (resp., s_2 and g_2) are the acculturation rates between F_1 and H (resp., F_2 and H). The probability density function p is the normalised conversion rate between F_1 and F_2 such that

$$
(H_p)\begin{cases}
p \in C^1(\mathbb{R}_+), \qquad 0 \le p(z) \le 1 \;\; \forall z \in \mathbb{R}_+, \qquad \dfrac{d}{dz}p(z) \ge 0 \;\; \forall z \in \mathbb{R}_+, \\
\phi : z \mapsto p(z)z \text{ is Lipschitz continuous with a Lipschitz constant } C_{Lip}.
\end{cases}
$$

For example, one may consider p of the form

$$
p(F) = p_m(F;F_c) = \frac{F^m}{F^m + F_c^m},
\tag{4}
$$

where m is a positive integer and F_c is a switching value for the conversion between F_1 and F_2, a density threshold at which the probability of remaining sedentary and migrating is equal. Finally, $\dfrac{1}{\varepsilon}$ is the speed of conversion between F_1 and F_2.

In this paper we study the system (3) under the additional assumption that $s_1 = s_2 =: s$, $g_1 = g_2 =: g$, $d_F =: d$ and $d_H = 1$, namely,

$$
\begin{cases}
F_{1t} = (1-F)F_1 + sF_1 H - \dfrac{1}{\varepsilon}\left(p(F)F_1 - (1-p(F))F_2\right), \\
F_{2t} = d\Delta F_2 + (1-F)F_2 + sF_2 H + \dfrac{1}{\varepsilon}\left(p(F)F_1 - (1-p(F))F_2\right), \\
H_t = \Delta H + b(1-H)H - gFH
\end{cases}
\tag{5}
$$

in an open bounded domain Ω with the homogeneous Neumann boundary conditions

$$
\partial_\nu F_2 = \partial_\nu H = 0,
\tag{6}
$$

where ν is the outward normal vector on $\partial\Omega$ and with the nonnegative initial conditions

$$
(F_1, F_2, H)(\mathbf{x}, 0) = (F_{10}, F_{20}, H_0)(\mathbf{x}).
\tag{7}
$$

First, we show numerically the behaviour of a solution of the two-dimensional problem (5)-(7) with $p(F) = p_2(F;F_c)$ in $\Omega_L = \{\mathbf{x} = (x,y) \in \mathbb{R}^2 \,|\, 0 \le x \le L, 0 \le y \le L\}$, where F_{10} is compactly supported in Ω_L, $F_{20} \equiv 0$ and $H_0 \equiv 1$ in Ω_L except in the region where $F_{10} > 0$, as shown in Fig. 2(a). In this modelling setting, only sedentary farmers F_1 and hunter-gatherers H initially exist. However,

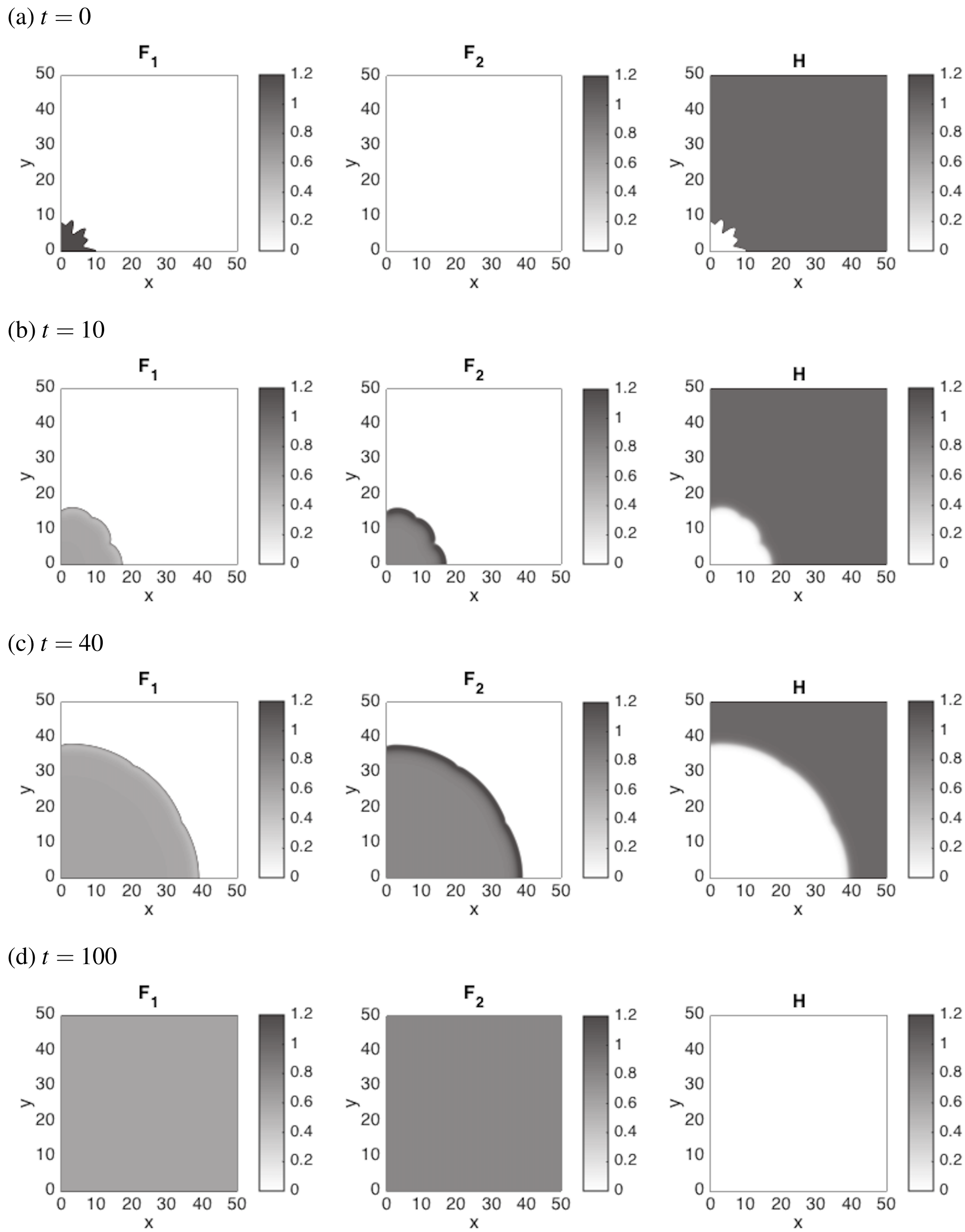

Figure 2: Spatial patterns of (F_1, F_2, H) of (5)-(7) in Ω_{50} where $s = 5.0$, $\varepsilon = 0.01$, $F_c = 0.5$, $d = 0.1$, $b = 1.0$ and $g = 4.0$. Here F_1 is rescaled by $3 \times F_1$.

even if F_2 does not initially exist, it eventually appears due to the conversion from F_1 as shown in Fig. 2(b). Then, the densities of the farming populations F_1 and F_2 expand uniformly throughout the domain, and farmers ($F = F_1 + F_2$) completely replace the population of hunter-gatherers who become extinct after large time. A characteristic property of (4) is that the expanding velocity of the total population of farmers F becomes slower as F_c increases, as shown in Figures 3 and 4. These figures suggest that the transition velocity can be slowed down by taking F_c suitably large. We hypothesise

(a) $F_c = 0.1$ (b) $F_c = 1$ (c) $F_c = 2$

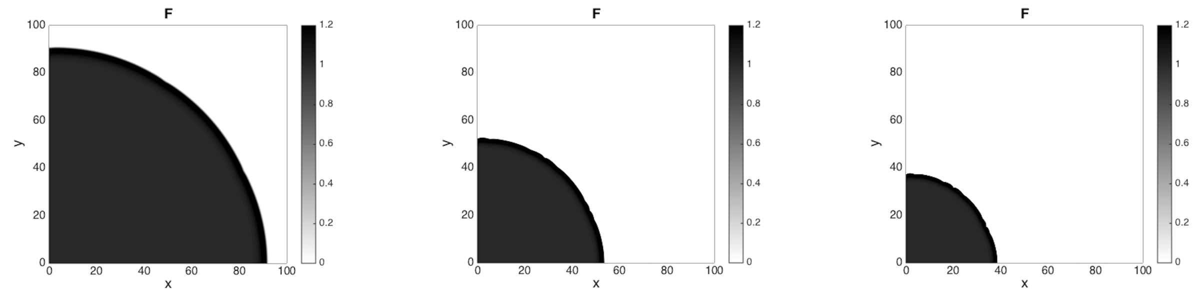

Figure 3: Spatial patterns of $F(x,y,80)$ of (5)-(7) in Ω_{100}, where the parameters are the same as in Fig. 2 except for F_c.

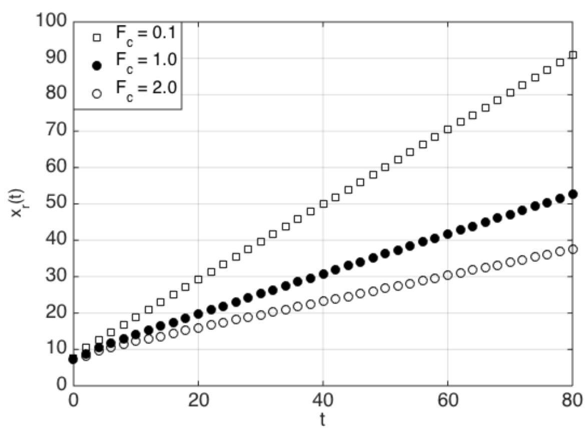

Figure 4: Average distance x_r between the front of F in Fig. 3 and the origin $(0,0)$.

that the threshold F_c, in a sense, reflects advancing farming technology in the Neolithic transition. Ecologically speaking, the elevated value of F_c can be interpreted as a certain level of development of farming and food-producing technology and associated sedentary lifestyle traits such as pottery making, domestication of various plants and animals (and related social changes, security, trading) that, as suggested by Childe in [8], can support a much larger population density than hunting and gathering and thus provide the basis for densely populated settlements. With all this said, the extent of how agriculture affected the decision-making to stay or migrate in the Neolithic transition can be modelled by the parameter F_c. We therefore propose the system (5) as a plausible dispersal model for describing the interaction between farmers and hunter-gatherers in the Neolithic transition.

The numerical results bring us to search for a rigorous setting. Hence, the purpose of this paper is to prove the existence and uniqueness of the global in time solution. However, the general theory on reaction-diffusion systems cannot be applied directly due to the lack of regularity. Therefore, a suitable well-posedness theory has to be developed.

From now on, we use the unknowns (u,v,w) instead of (F_1,F_2,H) and rewrite the system as

$$(\mathscr{P})\begin{cases} u_t=(1-u-v)u+suw-\dfrac{1}{\varepsilon}\left(p(u+v)u-(1-p(u+v))v\right) \quad \text{in } Q_T,\\ v_t=d\Delta v+(1-u-v)v+svw+\dfrac{1}{\varepsilon}\left(p(u+v)u-(1-p(u+v))v\right) \quad \text{in } Q_T,\\ w_t=\Delta w+b(1-w)w-g(u+v)w \quad \text{in } Q_T,\\ \partial_\nu v=\partial_\nu w=0 \quad \text{on } \Gamma_T,\\ (u(x,0),v(x,0),w(x,0))=(u_0,v_0,w_0) \quad \text{in } \Omega,\end{cases}$$

where Ω is an open bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary (e.g., $\partial\Omega\in C^2$), $Q_T=\Omega\times(0,T)$ and $\Gamma_T=\partial\Omega\times(0,T)$ for any $T>0$; ν is the unit normal vector on the boundary Γ_T pointing outward of Q_T. Moreover, we suppose that the initial functions u_0, v_0 and w_0 satisfy

$$(H_0)\quad u_0,v_0,w_0\in C(\overline{\Omega}) \text{ and } u_0,v_0\geq 0,\ 0\leq u_0+v_0\leq 1,\ 0\leq w_0\leq 1 \quad \text{in } \overline{\Omega}.$$

We will write

$$f_1(u,v,w)=(1-u-v)u+suw-\frac{1}{\varepsilon}\left(p(u+v)u-(1-p(u+v))v\right), \tag{8}$$

$$f_2(u,v,w)=(1-u-v)v+svw+\frac{1}{\varepsilon}\left(p(u+v)u-(1-p(u+v))v\right), \tag{9}$$

$$f_3(u,v,w)=b(1-w)w-g(u+v)w. \tag{10}$$

Definition 1.1. A triple of functions (u,v,w) is a weak solution of Problem $(\mathscr{P})$ if

i) $u\in C^{0,1}(0,T;L^\infty(\Omega))$,
$v,w\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$, $v_t,w_t\in L^2(0,T;(H^1(\Omega))')$,

ii) u,v,w satisfy

$$-\int_\Omega u_0\xi(0)\,\mathrm{d}x=\iint_{Q_T} f_1(u,v,w)\xi+u\xi_t\,\mathrm{d}x\,\mathrm{d}t \tag{11}$$

$$-\int_\Omega v_0\xi(0)\,\mathrm{d}x=\iint_{Q_T} dv\Delta\xi+f_2(u,v,w)\xi+v\xi_t\,\mathrm{d}x\,\mathrm{d}t \tag{12}$$

$$-\int_\Omega w_0\xi(0)\,\mathrm{d}x=\iint_{Q_T} w\Delta\xi+f_3(u,v,w)\xi+w\xi_t\,\mathrm{d}x\,\mathrm{d}t \tag{13}$$

for any test function $\xi\in C^{2,1}(\overline{Q}_T)$ such that $\xi(x,T)=0$ in Ω and $\partial_\nu\xi=0$ on $\partial\Omega\times[0,T]$.

The main theorem can be stated as follows:

Theorem 1.1. *Suppose that the hypotheses (H_p) and (H_0) are satisfied, then Problem $(\mathscr{P})$ possesses a unique weak solution (u,v,w) such that*

$$0\leq u,v\leq C_\varepsilon \qquad \text{and} \qquad 0\leq w\leq 1 \quad \text{in } \overline{Q}_T \tag{14}$$

where $C_\varepsilon=2(1+s+1/\varepsilon)$. Moreover, $(v,w)\in[W_p^{2,1}(\Omega\times(\delta,T))]^2$ for all $0<\delta<T$ and all $p\in[1,\infty)$.

The idea of the proof is to consider a related uniformly parabolic PDE/PDE problem, where we add diffusion to the equation for u,

$$(\mathscr{P}_a)\begin{cases} u_t = a\Delta u + (1-u-v)u + suw - \dfrac{1}{\varepsilon}\left(p(u+v)u - (1-p(u+v))v\right) \quad \text{in } Q_T, \\ v_t = d\Delta v + (1-u-v)v + svw + \dfrac{1}{\varepsilon}\left(p(u+v)u - (1-p(u+v))v\right) \quad \text{in } Q_T, \\ w_t = \Delta w + b(1-w)w - g(u+v)w \quad \text{in } Q_T, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0 \quad \text{on } \Gamma_T, \\ (u(x,0), v(x,0), w(x,0)) = (u_0, v_0, w_0) \quad \text{in } \Omega, \end{cases}$$

for some $a > 0$. We prove that Problem $(\mathscr{P}_a)$ possesses a unique classical solution $(u_a, v_a, w_a) \in [C^{2,1}(\overline{\Omega}\times(0,T]) \cap C(\overline{\Omega}\times[0,T])]^3$ which converges to the unique solution of Problem $(\mathscr{P})$ as $a \to 0$. The proof is thus based on a standard approach where the solution to the problem which we study is approximated by a sequence of regular solutions of related problems, the existence of which follows from the classical theory of semilinear equations. Other approaches such as direct fixed point method based upon analytic semigroups could be used; however, these approaches may bring other difficulties to be dealt with.

We conclude this introductory part with a remark that there exists an excessive amount of literature on reaction-diffusion equation. We mention the monographs by N. F. Britton [6], P. C. Fife [12], W.-M. Ni [26] and J. Smoller [27], and the articles [1, 10, 11, 18, 23, 24, 25].

2 Existence of a unique solution of Problem $(\mathscr{P}_a)$

Lemma 2.1. *Suppose that the hypotheses (H_p) and (H_0) are satisfied and suppose that (u_a, v_a, w_a) is the unique solution of Problem $(\mathscr{P}_a)$. Then (u_a, v_a, w_a) remains nonnegative for all times.*

Proof. Let us consider the auxiliary problem

$$(\mathscr{P}_a^+)\begin{cases} u_t = a\Delta u + (1-u-v)u + suw - \dfrac{1}{\varepsilon}\left(p(u+v)u - (1-p(u+v))v\right) \ \text{in } Q_T, \\ v_t = d\Delta v + (1-u-v)v + svw + \dfrac{1}{\varepsilon}\left(p(u+v)u^+ - (1-p(u+v))v\right) \ \text{in } Q_T, \\ w_t = \Delta w + b(1-w)w - g(u+v)w \ \text{in } Q_T, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0 \ \text{on } \Gamma_T, \\ (u(x,0), v(x,0), w(x,0)) = (u_0, v_0, w_0) \ \text{in } \Omega, \end{cases}$$

where $u^+ = \max\{u, 0\}$. Moreover, let us denote

$$\mathscr{L}_u(z) = z_t - a\Delta z - (1-z-v)z - szw + \frac{1}{\varepsilon}p(z+v)z - \frac{1}{\varepsilon}(1-p(z+v))v,$$
$$\mathscr{L}_v(z) = z_t - d\Delta z - (1-u-z)z - szw - \frac{1}{\varepsilon}p(u+z)u^+ + \frac{1}{\varepsilon}(1-p(u+z))z,$$

and

$$\mathscr{L}_w(z) = z_t - \Delta z - b(1-z)z + g(u+v)z. \tag{15}$$

Then, by the standard comparison principle applied subsequently to each equation, we obtain the non-negativeness of the solution. Indeed, on the one hand we have $\mathscr{L}_v(0) = -p(u)u^+/\varepsilon \leq 0$ which implies that $v \geq 0$ in $\overline{\Omega} \times [0,\infty)$. On the other hand, $\mathscr{L}_u(0) = -(1-p(v))v/\varepsilon \leq 0$ for $v \geq 0$. Therefore, $u \geq 0$ for each $(x,t) \in \overline{\Omega} \times [0,\infty)$. Finally, $\mathscr{L}_w(0) = 0$ implies $w \geq 0$ in $\overline{\Omega} \times [0,\infty)$.

Thus, for any nonnegative initial data $(u_0, v_0, w_0) \in \mathbb{R}^3_+$ the unique solution of Problem $(\mathscr{P}_a^+)$ is also the solution of Problem $(\mathscr{P}_a)$, which completes the proof. □

Theorem 2.2. *Let a and ε be positive constants and suppose that the hypotheses (H_p) and (H_0) are satisfied. Problem $(\mathscr{P}_a)$ admits a unique classical solution $(u_a, v_a, w_a) \in [C^{2,1}(\overline{\Omega} \times (0,T]) \cap C(\overline{\Omega} \times [0,T])]^3$ such that*

$$0 \leq u_a, v_a \leq C_\varepsilon \qquad \textit{and} \qquad 0 \leq w_a \leq 1 \quad \textit{in } \overline{Q}_T \tag{16}$$

where $C_\varepsilon = 2(1+s+1/\varepsilon)$.

Proof. The existence of the unique classical solution (u_a, v_a, w_a) of Problem $(\mathscr{P}_a)$ follows from the classical theory of semilinear equations, e.g., Proposition 7.3.2 [22] (p. 277). The positivity of the solution is proved in Lemma 2.1.

To find an upper bound for u_a and v_a let us consider the system $(\mathscr{P}^\#)$,

$$(\mathscr{P}^\#) \begin{cases} u_t - a\Delta u = F_1(u,v,w) \quad \text{in } Q_T, \\ v_t - d\Delta v = F_2(u,v,w) \quad \text{in } Q_T, \\ w_t - \Delta w = F_3(u,v,w) \quad \text{in } Q_T, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0 \quad \text{on } \Gamma_T, \\ (u(x,0), v(x,0), w(x,0)) = (u_0(x), v_0(x), w_0(x)) \quad \text{in } \Omega, \end{cases}$$

where

$$\begin{aligned} F_1(u,v,w) &= \max\{f_1(z_1,z_2,z_3);\ z_1 = u,\ 0 \leq z_2 \leq v,\ 0 \leq z_3 \leq w\}, \\ F_2(u,v,w) &= \max\{f_2(z_1,z_2,z_3);\ 0 \leq z_1 \leq u,\ z_2 = v,\ 0 \leq z_3 \leq w\}, \\ F_3(u,v,w) &= \max\{f_3(z_1,z_2,z_3);\ 0 \leq z_1 \leq u,\ 0 \leq z_2 \leq v,\ z_3 = w\}, \end{aligned}$$

and f_1, f_2 and f_3 are defined by (8), (9) and (10), respectively. We note that the nonlinearities f_1, f_2 and f_3 are continuous in their variables and that

$$F_k(u,v,w) \geq f_k(u,v,w), \quad k = 1,2,3. \tag{17}$$

Moreover, we see that

$$\begin{aligned} F_1(u,v,w) &= \max_{0 \leq z_2 \leq v,\ 0 \leq z_3 \leq w} \left\{ (1-u)u - uz_2 + suz_3 - \frac{1}{\varepsilon} p(u+z_2)(u+z_2) + \frac{z_2}{\varepsilon} \right\} \\ &\leq (1-u)u + suw + \frac{v}{\varepsilon}, \\ F_2(u,v,w) &= \max_{0 \leq z_1 \leq u,\ 0 \leq z_3 \leq w} \left\{ (1-v)v - z_1 v + svz_3 + \frac{1}{\varepsilon} p(z_1+v)(z_1+v) - \frac{v}{\varepsilon} \right\} \\ &\leq (1-v)v + svw + \frac{1}{\varepsilon} p(u+v)(u+v) - \frac{v}{\varepsilon}, \\ F_3(u,v,w) &= \max_{0 \leq z_1 \leq u,\ 0 \leq z_2 \leq v} \{ b(1-w)w - g(z_1+z_2)w \} = bw(1-w), \end{aligned}$$

and that the system $(\mathscr{P}^\dagger)$,

$$(\mathscr{P}^\dagger)\begin{cases} u_t - a\Delta u = \widetilde{F}_1(u,v,w) \quad \text{in } Q_T, \\ v_t - d\Delta v = \widetilde{F}_2(u,v,w) \quad \text{in } Q_T, \\ w_t - \Delta w = \widetilde{F}_3(w) \quad \text{in } Q_T, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0 \quad \text{on } \Gamma_T, \\ (u(x,0),v(x,0),w(x,0)) = (u_0(x),v_0(x),w_0(x)) \quad \text{in } \Omega, \end{cases}$$

where

$$\begin{aligned} \widetilde{F}_1(u,v,w) &= (1-u)u + suw + \frac{v}{\varepsilon}, \\ \widetilde{F}_2(u,v,w) &= (1-v)v + svw + \frac{1}{\varepsilon}p(u+v)(u+v) - \frac{v}{\varepsilon}, \\ \widetilde{F}_3(w) &= b(1-w)w, \end{aligned}$$

is a cooperative system since $\partial_v \widetilde{F}_1 = 1/\varepsilon \geq 0$, $\partial_w \widetilde{F}_1 = su \geq 0$, $\partial_u \widetilde{F}_2 = (p'(u+v)(u+v)+p(u+v))/\varepsilon \geq 0$, $\partial_w \widetilde{F}_2 = sv \geq 0$ and $\partial_u \widetilde{F}_3 = \partial_v \widetilde{F}_3 = 0$, see [20] on p. 168. Therefore it admits a comparison principle, [20] (Theorem D, p. 170). In particular, it follows from the comparison principle and (17), which in turn implies the inequalities

$$\widetilde{F}_k(u,v,w) \geq f_k(u,v,w), \quad k = 1,2,3, \tag{18}$$

that the solution (u_a,v_a,w_a) of Problem $(\mathscr{P}_a)$ is a lower solution for Problem $(\mathscr{P}^\dagger)$ in $\overline{\Omega}\times[0,\infty)$. Indeed, set

$$\begin{aligned} \mathscr{L}_1(u,v,w) &= \partial_t u - a\Delta u - \widetilde{F}_1(u,v,w), \\ \mathscr{L}_2(u,v,w) &= \partial_t v - d\Delta v - \widetilde{F}_2(u,v,w), \\ \mathscr{L}_3(u,v,w) &= \partial_t w - \Delta w - \widetilde{F}_3(w). \end{aligned}$$

Then,

$$\mathscr{L}_1(u_a,v_a,w_a) = \partial_t u_a - a\Delta u_a - \widetilde{F}_1(u_a,v_a,w_a) = f_1(u_a,v_a,w_a) - \widetilde{F}_1(u_a,v_a,w_a) \leq 0,$$

$$\mathscr{L}_2(u_a,v_a,w_a) = \partial_t v_a - d\Delta v_a - \widetilde{F}_2(u_a,v_a,w_a) = f_2(u_a,v_a,w_a) - \widetilde{F}_2(u_a,v_a,w_a) \leq 0,$$

and

$$\mathscr{L}_3(u_a,v_a,w_a) = \partial_t w_a - \Delta w_a - \widetilde{F}_3(w_a) = f_3(u_a,v_a,w_a) - \widetilde{F}_3(w_a) \leq 0.$$

by (18). Thus, we obtain

$$0 \leq u_a \leq u, \quad 0 \leq v_a \leq v \quad \text{and} \quad 0 \leq w_a \leq w \quad \text{in } \overline{\Omega}\times[0,\infty) \tag{19}$$

where (u,v,w) is the solution of Problem $(\mathscr{P}^\dagger)$. Therefore any upper bound for (u,v,w) is also the upper bound for (u_a,v_a,w_a).

Let (U,V,W) be the solution of the ODE problem

$$\begin{cases} U' = \widetilde{F}_1(U,V,W), \\ V' = \widetilde{F}_2(U,V,W), \\ W' = \widetilde{F}_3(W), \\ (U(0),V(0),W(0)) = (\|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^\infty(\Omega)}, \|w_0\|_{L^\infty(\Omega)}). \end{cases}$$

Again by the comparison principle, (U,V,W) is an upper solution for Problem $(\mathscr{P}^\dagger)$ (and consequently for Problem $(\mathscr{P}_a)$ by (19)). We remark that $0 \le W \le 1$. We set $Z = U+V$ and obtain the equation

$$
\begin{aligned}
Z' &= (1-U)U + sUW + \frac{V}{\varepsilon} + (1-V)V + sVW - \frac{V}{\varepsilon} + \frac{1}{\varepsilon}p(U+V)(U+V) \\
&= Z + sZW - (U^2+V^2) + \frac{1}{\varepsilon}p(Z)Z \\
&\le \left(1+s+\frac{1}{\varepsilon}\right)Z - \frac{1}{2}Z^2
\end{aligned}
$$

since $0 \le p \le 1$ and thanks to the elementary inequality $a^2+b^2 \ge (a+b)^2/2$ for $a,b \in \mathbb{R}_+$. Set $C_\varepsilon = 2(1+s+1/\varepsilon)$ and define $\overline{Z}$ as the solution of the initial value problem $(\mathscr{P}^\ddagger)$,

$$
(\mathscr{P}^\ddagger)\begin{cases} \overline{Z}' = (C_\varepsilon - \overline{Z})\overline{Z}/2, \\ \overline{Z}(0) = Z_0, \end{cases}
$$

where $Z_0 = \|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)} \in [0,2]$. Then Z is a lower solution for $(\mathscr{P}^\ddagger)$ and, by the comparison principle,

$$
0 \le Z = U+V \le \overline{Z} \le C_\varepsilon
$$

for each $t \ge 0$. Altogether,

$$
\begin{aligned}
&0 \le u_a \le u \le U \le C_\varepsilon, \\
&0 \le v_a \le v \le V \le C_\varepsilon, \\
&0 \le w_a \le w \le W \le 1
\end{aligned}
$$

in $\overline{\Omega} \times [0,\infty)$. □

Remark 2.1. Because of regularity results for parabolic problems (e.g. [21], Theorem IV.5.3) we have $(u_a,v_a,w_a) \in [C^{2+\alpha,(2+\alpha)/2}(\overline{Q}_T)]^3$ for all $\alpha \in (0,1)$ given that the initial data are sufficiently smooth.

Remark 2.2. We note that the construction of (F_1,F_2,F_3) of $(\mathscr{P}^\#)$ follows the classical approach of J. Smoller [27], where (F_1,F_2,F_3) is called the maximal vector field associated to the vector field (f_1,f_2,f_3). The solution of the ODE problem associated to $(\mathscr{P}^\#)$ is then shown to be an upper solution for Problem $(\mathscr{P}_a)$ by the comparison Theorem 14.16 in [27]. Since it is impossible to write (F_1,F_2,F_3) explicitly in our case, we cannot use this vector field directly.

Remark 2.3. The upper bound for w_a can be also obtained immediately by the comparison principle since $\mathscr{L}_w(1) \ge 0$ for $u_a, v_a \ge 0$ and $\mathscr{L}_w$ defined by (15).

Remark 2.4. For further use, let us mention that the classical solution (u_a,v_a,w_a) of Problem $(\mathscr{P}_a)$ satisfies the integral equalities

$$
-\int_\Omega u_0\xi(0)\,\mathrm{d}x = \iint_{Q_T} au_a\Delta\xi + f_1(u_a,v_a,w_a)\xi + u\xi_t\,\mathrm{d}x\,\mathrm{d}t \tag{20}
$$

$$
-\int_\Omega v_0\xi(0)\,\mathrm{d}x = \iint_{Q_T} dv_a\Delta\xi + f_2(u_a,v_a,w_a)\xi + v\xi_t\,\mathrm{d}x\,\mathrm{d}t \tag{21}
$$

$$
-\int_\Omega w_0\xi(0)\,\mathrm{d}x = \iint_{Q_T} w_a\Delta\xi + f_3(u_a,v_a,w_a)\xi + w\xi_t\,\mathrm{d}x\,\mathrm{d}t \tag{22}
$$

for any test function $\xi \in C^{2,1}(\overline{Q}_T)$ such that $\xi(x,T) = 0$ in Ω and $\partial_\nu\xi = 0$ on $\partial\Omega \times [0,T]$.

3 Singular limit problem $(\mathscr{P}_a)$ as $a \to 0$ for $\varepsilon > 0$ fixed

First, we show some a-priori estimates. We always assume that the hypotheses (H_p) and (H_0) are satisfied.

Lemma 3.1. *There exists a positive constant C_0 independent of the diffusion coefficient a such that*

$$\iint_{Q_T} u_a^2 \,\mathrm{d}x\,\mathrm{d}t, \iint_{Q_T} v_a^2 \,\mathrm{d}x\,\mathrm{d}t \le C_0 \tag{23}$$

Proof. The function $z = u_a + v_a$ satisfies the equation

$$z_t = a\Delta u_a + d\Delta v_a + (1-z)z + sw_a z.$$

We integrate it in space and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega z\,\mathrm{d}x = \int_\Omega ((1-z)z + sw_a z)\,\mathrm{d}x \le (1+s)\int_\Omega z\,\mathrm{d}x - \int_\Omega z^2\,\mathrm{d}x \tag{24}$$

$$\le (1+s)\int_\Omega z\,\mathrm{d}x - \frac{1}{|\Omega|}\left(\int_\Omega z\right)^2 \tag{25}$$

where we have used the homogeneous Neumann boundary conditions, the uniform bound $0 \le w_a \le 1$ and the Hölder inequality $(\int z)^2 \le |\Omega| \int z^2$. Therefore, in view of (25), $y(t) = \int_\Omega z(\cdot,t)$ satisfies the equation

$$y' \le (C_{s,|\Omega|} - y)y/|\Omega|, \quad y(0) = y_0,$$

where $y_0 = \int_\Omega z_0 = \int_\Omega (u_0 + v_0) \le |\Omega|$ and $C_{s,|\Omega|} = (1+s)|\Omega|$. From this equation we deduce the uniform L^1 bound

$$\int_\Omega z(t) \le C_{s,|\Omega|}, \quad \forall t > 0.$$

Integrating (24) in time for $t \in (0,T]$ and using the above estimate give

$$\int_\Omega z(T) + \int_0^T \int_\Omega z^2 \le \int_\Omega z_0 + (1+s)\int_0^T \int_\Omega z \le |\Omega| + (1+s)TC_{s,|\Omega|} =: C_0,$$

which completes the proof. □

Lemma 3.2. *There exist positive constants C_1 and C_2 independent of the diffusion coefficient a (but C_1 depends on ε) such that*

$$a\iint_{Q_T} |\nabla u_a|^2 \,\mathrm{d}x\,\mathrm{d}t, \quad d\iint_{Q_T} |\nabla v_a|^2 \,\mathrm{d}x\,\mathrm{d}t \le C_1. \tag{26}$$

$$\iint_{Q_T} |\nabla w_a|^2 \,\mathrm{d}x\,\mathrm{d}t \le C_2. \tag{27}$$

Proof. Let us multiply the equations for u_a and v_a by u_a and v_a, respectively, and integrate in space. We obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u_a^2 + v_a^2) + a\int_\Omega |\nabla u_a|^2 + d\int_\Omega |\nabla v_a|^2 + \int_\Omega (u_a + v_a)(u_a^2 + v_a^2) \\
&+\frac{1}{\varepsilon}\int_\Omega \{p(u_a+v_a)u_a^2 + (1-p(u_a+v_a))v_a^2\} = \int_\Omega (1+sw_a)(u_a^2+v_a^2) + \frac{1}{\varepsilon}\int_\Omega u_a v_a \qquad (28)\\
&\le (1+s+\frac{1}{2\varepsilon})\int_\Omega (u_a^2 + v_a^2)
\end{aligned}$$

where we have used the integration by parts formula, the upper bound $w_a \leq 1$ and the Young inequality. In view of (23), integrating (28) in time gives (26).

Similarly, we deduce from the equation for w_a that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w_a^2 + \int_\Omega |\nabla w_a|^2 + g\int_\Omega (u_a+v_a)w_a^2 = b\int_\Omega (1-w_a)w_a^2 \leq b|\Omega|,$$

thanks to the upper bound $w_a \leq 1$. Integrating this inequalities in time gives the estimate (27). □

Remark 3.1. We remark that the estimate (26) for u_a is not uniform in a. By using (16) and the estimates on the gradients (26) and (27) we obtain, for example,

$$\begin{aligned}
\|(u_a)_t\|_{L^2(0,T;(H^1(\Omega))')} &:= \sup_{\|\varphi\|_{L^2(0,T;H^1(\Omega))}\leq 1} \left| \int_0^T \left\langle \frac{\partial u_a}{\partial t}, \varphi \right\rangle_{(H^1)'\times H^1} \mathrm{d}t \right| \\
&\leq \sup_{\|\varphi\|_{L^2(0,T;H^1(\Omega))}\leq 1} \Big\{ a\|\nabla u_a\|_{L^2(Q_T)}\|\nabla\varphi\|_{L^2(Q_T)} \\
&\qquad + \|f_1(u_a,v_a,w_a\|_{L^2(Q_T)}\|\varphi\|_{L^2(Q_T)} \Big\} \\
&\leq C(\varepsilon).
\end{aligned}$$

We deduce that $u_a, v_a, w_a \in \left\{\vartheta \in L^2(0,T;H^1(\Omega)),\ \vartheta_t \in L^2(0,T;(H^1(\Omega))')\right\}$ as the above calculations hold for v_a and w_a. As a consequence $\{(u_a, v_a w_a)\}_{a>0} \in [C(0,T;L^2(\Omega))]^3$.

The main tool that is used to prove the existence of the solution of Problem $(\mathscr{P})$ is the following Fréchet-Kolmogorov compactness theorem, e.g. [5], Theorem IV.25 on p. 72; the presented form below is taken from [9], Proposition 2.5.

Theorem 3.3 (Fréchet-Kolmogorov)**.** *Let $\mathscr{F}$ be a bounded subset of $L^p(Q_T)$ with $1 \leq p < \infty$. Assume that*

i) for any $\eta > 0$ and any subset $\omega \Subset Q_T$, there exists $\delta > 0$ ($\delta < dist(\omega, \partial Q_T)$) such that

$$\|f(x+\xi,t) - f(x,t)\|_{L^p(\omega)} + \|f(x,t+\tau) - f(x,t)\|_{L^p(\omega)} < \eta$$

for all ξ, τ and $f \in \mathscr{F}$ satisfying $|\xi| + |\tau| < \delta$.

ii) for any $\eta > 0$, there exists a subset $\omega \Subset Q_T$ such that

$$\|f\|_{L^p(Q_T\setminus\omega)} < \eta$$

for all $f \in \mathscr{F}$.

Then $\mathscr{F}$ is precompact in $L^p(Q_T)$.

Throughout the paper we will consider two subsets Ω_r and Ω_r' of Ω; in particular, for sufficiently small $r > 0$ we define $\Omega_r = \{x \in \Omega \,|\, B(x,2r) \subset \Omega\}$ and $\Omega_r' = \cup_{x\in\Omega_r} B(x,r)$, where $B(x,r)$ denotes the ball in $\mathbb{R}^N$ with centre x and radius r. We have $\Omega_r \subset \Omega_r' \subset \Omega$.

Lemma 3.4. *Let $r \in (0, \hat{r})$ for some $\hat{r} > 0$ sufficiently small. There exist positive constants C_4, C_5 and C_6 independent of the diffusion coefficient a such that*

$$\int_0^{T-\tau} \int_{\Omega_r} (u_a(x,t+\tau) - u_a(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \le C_4 \tau, \tag{29}$$

$$\int_0^{T-\tau} \int_{\Omega_r} (v_a(x,t+\tau) - v_a(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \le C_5 \tau, \tag{30}$$

$$\int_0^{T-\tau} \int_{\Omega_r} (w_a(x,t+\tau) - w_a(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \le C_6 \tau \tag{31}$$

for all $\tau \in (0, T)$.

Proof. We prove (29) since (30) and (31) can be proved analogously. We can write

$$\begin{aligned}
&\int_0^{T-\tau} \int_{\Omega_r} (u_a(x,t+\tau) - u_a(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le \int_0^{T-\tau} \int_{\Omega} (u_a(x,t+\tau) - u_a(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_0^{T-\tau} \int_{\Omega} (u_a(x,t+\tau) - u_a(x,t)) \left(\int_t^{t+\tau} \partial_t u_a(x,t') \,\mathrm{d}t' \right) \mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_0^{T-\tau} \int_{\Omega} (u_a(x,t+\tau) - u_a(x,t)) \left(\int_0^{\tau} \partial_t u_a(x,t+t') \,\mathrm{d}t' \right) \mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_0^{\tau} \int_0^{T-\tau} \int_{\Omega} (u_a(x,t+\tau) - u_a(x,t)) \partial_t u_a(x,t+t') \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}t' \\
&\qquad = \int_0^{\tau} \int_0^{T-\tau} \int_{\Omega} \big\{ a(u_a(x,t+\tau) - u_a(x,t)) \Delta u_a(x,t+t') \\
&\qquad\qquad + (u_a(x,t+\tau) - u_a(x,t)) f_1(u_a, v_a, w_a)(x,t+t') \big\} \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}t' \\
&\qquad = I_1 + I_2.
\end{aligned}$$

The first integral with the Laplacian can be estimated in the following way,

$$\begin{aligned}
I_1 &:= a \int_0^{\tau} \int_0^{T-\tau} \int_{\Omega} (u_a(x,t+\tau) - u_a(x,t)) \Delta u_a(x,t+t') \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}t' \\
&= -a \int_0^{\tau} \int_0^{T-\tau} \int_{\Omega} \nabla [u_a(x,t+\tau) - u_a(x,t)] \cdot \nabla u_a(x,t+t') \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}t' \\
&\le 2a\tau \int_0^T \int_{\Omega} |\nabla u_a(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\le 2C_1 \tau
\end{aligned}$$

where we have used the Hölder inequality and (26). By using the bounds (16) uniform in the diffusion coefficient a we can easily find a positive constant $C = C(C_\varepsilon, T, |\Omega|)$ such that

$$I_2 := \int_0^{\tau} \int_0^{T-\tau} \int_{\Omega} (u_a(x,t+\tau) - u_a(x,t)) f_1(u_a, v_a, w_a)(x,t+t') \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}t'$$

$$\leq C\tau.$$

We deduce the inequality (29) from both estimates for I_1 and I_2. $\square$

Lemma 3.5. *For each $r \in (0,\hat{r})$ and $\hat{r} > 0$ sufficiently small, it holds that*

$$\int_0^T \int_{\Omega_r} (u_a(x+\xi,t) - u_a(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \leq \frac{C_1}{a}|\xi|^2, \tag{32}$$

$$\int_0^T \int_{\Omega_r} (v_a(x+\xi,t) - v_a(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \leq \frac{C_1}{d}|\xi|^2, \tag{33}$$

and

$$\int_0^T \int_{\Omega_r} (w_a(x+\xi,t) - w_a(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \leq C_2|\xi|^2 \tag{34}$$

for all $\xi \in \mathbb{R}^N$, $|\xi| \leq r$ where the constants C_1 and C_2 are given by (26) and (27).

Proof. In the case of (32) we can write

$$\begin{aligned}
a\int_0^T \int_{\Omega_r} &(u_a(x+\xi,t) - u_a(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \\
&= a\int_0^T \int_{\Omega_r} \left(\int_0^1 \nabla u_a(x+\theta\xi,t)\cdot\xi \,\mathrm{d}\theta\right)^2 \mathrm{d}x\,\mathrm{d}t \\
&\leq a|\xi|^2 \int_0^1 \int_0^T \int_{\Omega_r} |\nabla u_a(x+\theta\xi,t)|^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\theta \\
&\leq a|\xi|^2 \int_0^T \int_{\Omega_r'} |\nabla u_a(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\leq C_1|\xi|^2
\end{aligned}$$

due to (26). Analogously we prove (33) and (34). $\square$

Corollary 3.6. *The sequences $\{v_a\}_{a>0}$ and $\{w_a\}_{a>0}$ are relatively compact in $L^2(Q_T)$.*

Proof. We see from (30) and (33) (resp., (31) and (34)) that differences of space and time translates of v_a (resp., w_a) tend to zero uniformly in a in L^2 topology as the translation parameter tends to zero.

Moreover, in view of (16) we have that

$$\int_{T-\tau}^T \int_\Omega (v_a(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \leq C_\varepsilon^2|\Omega|\tau \quad \text{and} \quad \int_0^T \int_{\Omega\setminus\Omega_r} (v_a(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \leq 2C_\varepsilon^2 T|\partial\Omega|r,$$

and

$$\int_{T-\tau}^T \int_\Omega (w_a(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \leq |\Omega|\tau \quad \text{and} \quad \int_0^T \int_{\Omega\setminus\Omega_r} (w_a(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \leq 2T|\partial\Omega|r.$$

which implies that the hypothesis *ii)* of the Fréchet-Kolmogorov Theorem 3.3 is satisfied. Applying this theorem to $\{v_a\}_{a>0}$ and $\{w_a\}_{a>0}$ implies that the sequences $\{v_a\}_{a>0}$ and $\{w_a\}_{a>0}$ are relatively compact in $L^2(Q_T)$. $\square$

Because of the dependence of (32) on the parameter a, we are not able to control L^2-differences of space translates in the case of the sequence $\{u_a\}_{a>0}$ uniformly in a and to deduce the relative compactness of $\{u_a\}_{a>0}$ in $L^2(Q_T)$ as above. Nevertheless, the desired result can be obtained through the L^1 estimates on differences of space translates.

Lemma 3.7. *For each $r \in (0,\hat{r})$ and $\hat{r} > 0$ sufficiently small, there exists a positive function $\rho(\xi)$ such that $\rho(\xi) \to 0$ uniformly in a as $|\xi| \to 0$ and*

$$\int_0^T \int_{\Omega_r} |u_a(x+\xi,t) - u_a(x,t)| \,\mathrm{d}x\,\mathrm{d}t \le \rho(\xi) \tag{35}$$

for all $\xi \in \mathbb{R}^N$, $|\xi| \le r$.

Proof. In this proof only we will write shortly $u_\xi = u_a(x+\xi,t)$, $v_\xi = v_a(x+\xi,t)$, $w_\xi = w_a(x+\xi,t)$, $u = u_a(x,t)$, $v = v_a(x,t)$ and $w = w_a(x,t)$ and we will use the following notation:

$$\begin{aligned}
\hat{u} &= u_a(x+\xi,t) - u_a(x,t) = u_\xi - u, & \bar{u} &= u_a(x+\xi,t) + u_a(x,t) = u_\xi + u,\\
\hat{v} &= v_a(x+\xi,t) - v_a(x,t) = v_\xi - v, & \bar{v} &= v_a(x+\xi,t) + v_a(x,t) = v_\xi + v,\\
\hat{w} &= w_a(x+\xi,t) - w_a(x,t) = w_\xi - w, & \bar{w} &= w_a(x+\xi,t) + w_a(x,t) = w_\xi + w.
\end{aligned}$$

We recall that $0 \le \bar{u}, \bar{v} \le 2C_\varepsilon$ and $0 \le \bar{w} \le 2$ by (16). We will also consider a smooth convex function $m : \mathbb{R} \to \mathbb{R}_+$ such that $m \ge 0$, $m(0) = 0$ and $m(r) = |r| - 1/2$ for $|r| > 1$, and define for $\alpha > 0$ approximations of m by

$$m_\alpha(r) = \alpha m\left(\frac{r}{\alpha}\right).$$

Then, m_α satisfies

$$m_\alpha(r) \to |r| \quad \text{and} \quad m'_\alpha(r) \to \operatorname{sgn}(r)$$

as $\alpha \to 0$. Furthermore, we define a function μ such that

$$\mu \in C_0^\infty(\Omega'_r),\;\; 0 \le \mu(x) \le 1 \;\text{ in }\; \Omega'_r,\;\; \mu(x) = 1 \;\text{ in }\; \Omega_r \quad \text{and}$$
$$|\nabla\mu|, |\Delta\mu| \le C(r).$$

First, we multiply the equation for $\hat{u}$, i.e.,

$$\partial_t \hat{u} = a\Delta\hat{u} + f_1(u_\xi, v_\xi, w_\xi) - f_1(u,v,w)$$

by $\mu m'_\alpha(\hat{u})$ and integrate in space. We obtain

$$\begin{aligned}
\int_{\Omega'_r} \partial_t \hat{u} \mu m'_\alpha(\hat{u}) \,\mathrm{d}x = a &\int_{\Omega'_r} \Delta\hat{u}(\mu m'_\alpha(\hat{u})) \,\mathrm{d}x\\
&+ \int_{\Omega'_r} (f_1(u_\xi, v_\xi, w_\xi) - f_1(u,v,w))\mu m'_\alpha(\hat{u}) \,\mathrm{d}x
\end{aligned}$$

and, after integration by parts,

$$\begin{aligned}\int_{\Omega_r'} \partial_t m_\alpha(\hat{u})\mu \,\mathrm{d}x &= -a\int_{\Omega_r'} \nabla\hat{u}\cdot\nabla(\mu m_\alpha'(\hat{u}))\,\mathrm{d}x \\ &\quad + \int_{\Omega_r'} (f_1(u_\xi,v_\xi,w_\xi)-f_1(u,v,w))\mu m_\alpha'(\hat{u})\,\mathrm{d}x \\ &= -a\int_{\Omega_r'} m_\alpha'(\hat{u})\nabla\hat{u}\cdot\nabla\mu\,\mathrm{d}x - a\int_{\Omega_r'}\mu m_\alpha''(\hat{u})|\nabla\hat{u}|^2\,\mathrm{d}x \\ &\quad + \int_{\Omega_r'} (f_1(u_\xi,v_\xi,w_\xi)-f_1(u,v,w))\mu m_\alpha'(\hat{u})\,\mathrm{d}x \\ &\le -a\int_{\Omega_r'} \nabla m_\alpha(\hat{u})\cdot\nabla\mu\,\mathrm{d}x \\ &\quad + \int_{\Omega_r'} (f_1(u_\xi,v_\xi,w_\xi)-f_1(u,v,w))\mu m_\alpha'(\hat{u})\,\mathrm{d}x\end{aligned}$$

where we have applied the fact that $m_\alpha'' \ge 0$. Integration by parts once again yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_r'} m_\alpha(\hat{u})\mu\,\mathrm{d}x &\le a\int_{\Omega_r'} m_\alpha(\hat{u})\Delta\mu\,\mathrm{d}x \\ &\quad + \int_{\Omega_r'} (f_1(u_\xi,v_\xi,w_\xi)-f_1(u,v,w))\mu m_\alpha'(\hat{u})\,\mathrm{d}x.\end{aligned}$$

Finally, we can integrate in time to deduce that

$$\begin{aligned}\int_{\Omega_r'} m_\alpha(\hat{u})(t)\mu\,\mathrm{d}x &\le \int_{\Omega_r'} m_\alpha(\hat{u})(0)\mu\,\mathrm{d}x + a\int_0^t\int_{\Omega_r'} m_\alpha(\hat{u})\Delta\mu\,\mathrm{d}x\,\mathrm{d}t' \\ &\quad + \int_0^t\int_{\Omega_r'} (f_1(u_\xi,v_\xi,w_\xi)-f_1(u,v,w))\mu m_\alpha'(\hat{u})\,\mathrm{d}x\,\mathrm{d}t'.\end{aligned}$$

The Lebesgue Dominated Convergence Theorem allows us to pass to the limit $\alpha\to 0$ in the last inequality to obtain

$$\begin{aligned}\int_{\Omega_r'} |\hat{u}(t)|\mu\,\mathrm{d}x &\le \int_{\Omega_r'} |\hat{u}(0)|\mu\,\mathrm{d}x + a\int_0^t\int_{\Omega_r'} |\hat{u}|\Delta\mu\,\mathrm{d}x\,\mathrm{d}t' \\ &\quad + \int_0^t\int_{\Omega_r'} (f_1(u_\xi,v_\xi,w_\xi)-f_1(u,v,w))\mu\,\mathrm{sgn}(\hat{u})\,\mathrm{d}x\,\mathrm{d}t'\end{aligned}$$

where

$$\mathrm{sgn}(z) = \begin{cases} 1 & \text{if } z>0, \\ 0 & \text{if } z=0, \\ -1 & \text{if } z<0. \end{cases}$$

We apply the Hölder inequality and (32) to estimate the integral containing $\Delta\mu$ so that

$$\begin{aligned}a\int_0^t\int_{\Omega_r'} |\hat{u}|\Delta\mu\,\mathrm{d}x\,\mathrm{d}t' &\le \sqrt{a}\left(a\int_0^t\int_{\Omega_r'}|\hat{u}|^2\,\mathrm{d}x\,\mathrm{d}t'\right)^{1/2}\left(\int_0^t\int_{\Omega_r'}(\Delta\mu)^2\,\mathrm{d}x\,\mathrm{d}t'\right)^{1/2} \\ &\le \sqrt{a^*C_1T}\,\|\Delta\mu\|_{L^2(\Omega_r')}|\xi|\end{aligned}$$

where we have assumed without loss of generality that $a \leq a^*$ for some $a^* > 0$[1]. Whence, we obtain

$$\int_{\Omega'_r} |\hat{u}(t)|\mu \,\mathrm{d}x \leq \int_{\Omega'_r} |\hat{u}(0)|\mu \,\mathrm{d}x + C_7(r)|\xi| \\ + \int_0^t \int_{\Omega'_r} (f_1(u_\xi, v_\xi, w_\xi) - f_1(u,v,w))\mu \,\mathrm{sgn}(\hat{u}) \,\mathrm{d}x \,\mathrm{d}t' \tag{36}$$

where $C_7 = \sqrt{a^* C_1 T} \|\Delta\mu\|_{L^2(\Omega'_r)}$.

Analogously, by using (33) and (34) we can prove that

$$\int_{\Omega'_r} |\hat{v}(t)|\mu \,\mathrm{d}x \leq \int_{\Omega'_r} |\hat{v}(0)|\mu \,\mathrm{d}x + C_8(r)|\xi| \\ + \int_0^t \int_{\Omega'_r} (f_2(u_\xi, v_\xi, w_\xi) - f_2(u,v,w))\mu \,\mathrm{sgn}(\hat{v}) \,\mathrm{d}x \,\mathrm{d}t' \tag{37}$$

and

$$\int_{\Omega'_r} |\hat{w}(t)|\mu \,\mathrm{d}x \leq \int_{\Omega'_r} |\hat{w}(0)|\mu \,\mathrm{d}x + C_9(r)|\xi| \\ + \int_0^t \int_{\Omega'_r} (f_3(u_\xi, v_\xi, w_\xi) - f_3(u,v,w))\mu \,\mathrm{sgn}(\hat{w}) \,\mathrm{d}x \,\mathrm{d}t' \tag{38}$$

where $C_8 = \sqrt{dC_1 T} \|\Delta\mu\|_{L^2(\Omega'_r)}$ and $C_9 = \sqrt{C_2 T} \|\Delta\mu\|_{L^2(\Omega'_r)}$.

To estimate the nonlinearities, we first remark that

$$\begin{aligned} f_1(u_\xi, v_\xi, w_\xi) - f_1(u,v,w) &= \hat{u} - \hat{u}\bar{u} - (u_\xi v_\xi - uv) + s(u_\xi w_\xi - uw) + \frac{1}{\varepsilon}\hat{v} \\ &\quad - \frac{1}{\varepsilon}(\phi(u_\xi + v_\xi) - \phi(u+v)) \\ &= \hat{u} - \hat{u}\bar{u} - \frac{1}{2}\bar{u}\hat{v} - \frac{1}{2}\hat{u}\bar{v} + \frac{s}{2}\bar{u}\hat{w} + \frac{s}{2}\hat{u}\bar{w} + \frac{1}{\varepsilon}\hat{v} \\ &\quad - \frac{1}{\varepsilon}(\phi(u_\xi + v_\xi) - \phi(u+v)) \end{aligned}$$

where we have used the notation $\phi(z) = p(z)z$ and a trivial expansion

$$ab - cd = \frac{1}{2}(a+c)(b-d) + \frac{1}{2}(a-c)(b+d)$$

for any real numbers a, b, c and d. Similarly we derive

$$\begin{aligned} f_2(u_\xi, v_\xi, w_\xi) - f_2(u,v,w) &= \hat{v} - \hat{v}\bar{v} - \frac{1}{2}\bar{u}\hat{v} - \frac{1}{2}\hat{u}\bar{v} + \frac{s}{2}\bar{v}\hat{w} + \frac{s}{2}\hat{v}\bar{w} - \frac{1}{\varepsilon}\hat{v} \\ &\quad + \frac{1}{\varepsilon}(\phi(u_\xi + v_\xi) - \phi(u+v)), \\ f_3(u_\xi, v_\xi, w_\xi) - f_3(u,v,w) &= b\hat{w} - b\hat{w}\bar{w} - \frac{g}{2}\bar{u}\hat{w} - \frac{g}{2}\hat{u}\bar{w} - \frac{g}{2}\bar{v}\hat{w} - \frac{g}{2}\hat{v}\bar{w}. \end{aligned}$$

[1] We consider the problem with vanishing diffusion $a \to 0$.

By applying the assumption of Lipschitz continuity of ϕ we obtain

$$
\begin{aligned}
J_1 :=& \int_0^t \int_{\Omega_r'} \operatorname{sgn}(\hat{u}) \left(f_1(u_\xi, v_\xi, w_\xi) - f_1(u,v,w) \right) \mu \, \mathrm{d}x \, \mathrm{d}t' \\
\leq& \int_0^t \int_{\Omega_r'} \Big\{ |\hat{u}| - \bar{u}|\hat{u}| - \frac{1}{2} \operatorname{sgn}(\hat{u}) \bar{u}\hat{v} - \frac{1}{2} |\hat{u}|\bar{v} + \frac{s}{2} |\hat{u}|\bar{w} + \frac{s}{2} \operatorname{sgn}(\hat{u}) \bar{u}\hat{w} \\
& + \frac{1}{\varepsilon} \operatorname{sgn}(\hat{u})\hat{v} + \frac{C_{Lip}}{\varepsilon} |\hat{u} + \hat{v}| \Big\} \mu \, \mathrm{d}x \, \mathrm{d}t' \\
\leq& \int_0^t \int_{\Omega_r'} \Big\{ |\hat{u}| - \frac{1}{2} \operatorname{sgn}(\hat{u}) \bar{u}\hat{v} - \frac{1}{2} |\hat{u}|\bar{v} + s|\hat{u}| + sC_\varepsilon |\hat{w}| + \frac{1}{\varepsilon} \operatorname{sgn}(\hat{u})\hat{v} \\
& + \frac{C_{Lip}}{\varepsilon} |\hat{u} + \hat{v}| \Big\} \mu \, \mathrm{d}x \, \mathrm{d}t',
\end{aligned}
$$

since $\bar{u}|\hat{u}| \geq 0$, $0 \leq \bar{w} \leq 2$ and $0 \leq \bar{u} \leq 2C_\varepsilon$. Similarly,

$$
\begin{aligned}
J_2 :=& \int_0^t \int_{\Omega_r'} \operatorname{sgn}(\hat{v}) \left(f_2(u_\xi, v_\xi, w_\xi) - f_2(u,v,w) \right) \mu \, \mathrm{d}x \, \mathrm{d}t' \\
\leq& \int_0^t \int_{\Omega_r'} \Big\{ |\hat{v}| - \frac{1}{2} \operatorname{sgn}(\hat{v}) \bar{v}\hat{u} - \frac{1}{2} |\hat{v}|\bar{u} + s|\hat{v}| + sC_\varepsilon |\hat{w}| - \frac{1}{\varepsilon} |\hat{v}| \\
& + \frac{C_{Lip}}{\varepsilon} |\hat{u} + \hat{v}| \Big\} \mu \, \mathrm{d}x \, \mathrm{d}t'
\end{aligned}
$$

and

$$
\begin{aligned}
J_3 :=& \int_0^t \int_{\Omega_r'} \operatorname{sgn}(\hat{w}) \left(f_3(u_\xi, v_\xi, w_\xi) - f_3(u,v,w) \right) \mu \, \mathrm{d}x \, \mathrm{d}t' \\
=& \int_0^t \int_{\Omega_r'} \Big\{ b|\hat{w}| - b\bar{w}|\hat{w}| - \frac{g}{2} \operatorname{sgn}(\hat{w}) \bar{w}\hat{u} - \frac{g}{2} |\hat{w}|\bar{u} - \frac{g}{2} \operatorname{sgn}(\hat{w}) \bar{w}\hat{v} \\
& - \frac{g}{2} |\hat{w}|\bar{v} \Big\} \mu \, \mathrm{d}x \, \mathrm{d}t' \\
\leq& \int_0^t \int_{\Omega_r'} \{ b|\hat{w}| + g|\hat{u}| + g|\hat{v}| \} \mu \, \mathrm{d}x \, \mathrm{d}t'.
\end{aligned}
$$

Altogether,

$$
\begin{aligned}
J_1 + J_2 + J_3 \leq& \int_0^t \int_{\Omega_r'} \Big\{ |\hat{u}| + |\hat{v}| + b|\hat{w}| - \frac{1}{2} \bar{u}(|\hat{v}| + \operatorname{sgn}(\hat{u})\hat{v}) - \frac{1}{2} \bar{v}(|\hat{u}| + \operatorname{sgn}(\hat{v})\hat{u}) \\
& - \frac{1}{\varepsilon} (|\hat{v}| - \operatorname{sgn}(\hat{u})\hat{v}) + (s+g)|\hat{u}| + (s+g)|\hat{v}| \\
& + 2\frac{C_{Lip}}{\varepsilon} (|\hat{u}| + |\hat{v}|) + 2sC_\varepsilon |\hat{w}| \Big\} \mu \, \mathrm{d}x \, \mathrm{d}t' \\
\leq& \max \left\{ 1 + s + g + 2\frac{C_{Lip}}{\varepsilon}, \, b + 2sC_\varepsilon \right\} \int_0^t \int_{\Omega_r'} (|\hat{u}| + |\hat{v}| + |\hat{w}|) \mu \, \mathrm{d}x \, \mathrm{d}t'
\end{aligned}
$$

since $\bar{u}(|\hat{v}| + \mathrm{sgn}(\hat{u})\hat{v}) \geq 0$, $\bar{v}(|\hat{u}| + \mathrm{sgn}(\hat{v})\hat{u}) \geq 0$ and $|\hat{v}| - \mathrm{sgn}(\hat{u})\hat{v} \geq 0$. Hence, we find a positive constant $C_{10}^{\varepsilon} = \max\left\{1+s+g+2\frac{C_{Lip}}{\varepsilon},\ b+2sC_{\varepsilon}\right\}$ such that

$$J_1 + J_2 + J_3 \leq C_{10}^{\varepsilon} \int_0^t \int_{\Omega_r'} (|\hat{u}| + |\hat{v}| + |\hat{w}|)\mu \,\mathrm{d}x\,\mathrm{d}t'. \tag{39}$$

By adding all the estimates (36)-(39) together we deduce that

$$\begin{aligned}\int_{\Omega_r'} (|\hat{u}(t)| + |\hat{v}(t)| + |\hat{w}(t)|)\mu \,\mathrm{d}x \leq & \int_{\Omega_r'} (|\hat{u}(0)| + |\hat{v}(0)| + |\hat{w}(0)|)\mu \,\mathrm{d}x \\ & + (C_7 + C_8 + C_9)|\xi| + C_{10}^{\varepsilon} \int_0^t \int_{\Omega_r'} (|\hat{u}| + |\hat{v}| + |\hat{w}|)\mu \,\mathrm{d}x\,\mathrm{d}t'\end{aligned}$$

for $t > 0$. The Gronwall inequality implies that

$$\begin{aligned}\int_{\Omega_r'} (|\hat{u}(t)| + |\hat{v}(t)| + |\hat{w}(t)|)\mu \,\mathrm{d}x \leq \Big(& \int_{\Omega_r'} (|\hat{u}(0)| + |\hat{v}(0)| + |\hat{w}(0)|)\mu \,\mathrm{d}x \\ & + (C_7 + C_8 + C_9)|\xi| \Big) e^{C_{10}^{\varepsilon} T}.\end{aligned}$$

Thanks to the uniform boundedness of the initial data in Ω_r', there exists a positive function ω such that $\omega(\xi) \to 0$ as $\xi \to 0$ and

$$\int_{\Omega_r'} (|\hat{u}(0)| + |\hat{v}(0)| + |\hat{w}(0)|)\mu \,\mathrm{d}x \leq \omega(\xi)$$

for each $x \in \Omega_r'$ and $\xi \in \mathbb{R}^N$ such that $|\xi| \leq r$. Thus, we deduce the existence of a function $\rho(\xi)$ satisfying

$$\begin{aligned}\int_0^T \int_{\Omega_r} |\hat{u}| + |\hat{v}| + |\hat{w}| \,\mathrm{d}x\,\mathrm{d}t & \leq \int_0^T \int_{\Omega_r'} (|\hat{u}| + |\hat{v}| + |\hat{w}|)\mu \,\mathrm{d}x\,\mathrm{d}t \\ & \leq C(\omega(\xi) + |\xi|) e^{C_{10}^{\varepsilon} T} =: \rho(\xi)\end{aligned}$$

and $\rho(\xi) \to 0$ for $|\xi| \to 0$. □

Corollary 3.8. *The sequence $\{u_a\}_{a>0}$ is relatively compact in $L^2(Q_T)$.*

Proof. We deduce from (29) that the L^2-differences of time translates of u_a tend to zero as the translation parameter τ tends to zero. As for differences of space translates, we obtain

$$\begin{aligned}\int_{\Omega_r} (u_a(x+\xi,t) - u_a(x,t))^2 \,\mathrm{d}x & \leq 2 \sup_{\Omega_r} |u_a(x,t)| \int_{\Omega_r} |u_a(x+\xi,t) - u_a(x,t)| \,\mathrm{d}x \\ & \leq 2C_{\varepsilon} \int_{\Omega_r} |u_a(x+\xi,t) - u_a(x,t)| \,\mathrm{d}x\end{aligned}$$

for any $t \in (0,T]$, where we have used the L^{∞}-bound (16) for u_a. Integration in time and the inequality (35) from Lemma 3.7 yield

$$\int_0^T \int_{\Omega_r} (u_a(x+\xi,t) - u_a(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \leq 2C_{\varepsilon}\rho(\xi),$$

where the right-hand-side tends to zero uniformly in a as $\xi \to 0$. Moreover, similarly as in Corollary 3.6, we deduce that

$$\int_{T-\tau}^{T}\int_{\Omega}(u_a(x,t))^2\,\mathrm{d}x\,\mathrm{d}t \le C_\varepsilon^2|\Omega|\tau \quad \text{and} \quad \int_{0}^{T}\int_{\Omega\setminus\Omega_r}(u_a(x,t))^2\,\mathrm{d}x\,\mathrm{d}t \le 2C_\varepsilon^2 T|\partial\Omega|r.$$

The Fréchet-Kolmogorov Theorem 3.3 applied to $\{u_a\}_{a>0}$ allows us to conclude the relative compactness of the sequence $\{u_a\}_{a>0}$ in $L^2(Q_T)$. □

Corollary 3.9. *There exist a subsequence $\{(u_{a_n},v_{a_n},w_{a_n})\}_{a_n>0}$ and functions $u,v,w \in L^\infty(Q_T)$ such that*

$$\begin{aligned}&(u_{a_n},v_{a_n},w_{a_n}) \to (u,v,w) \text{ strongly in } [L^2(Q_T)]^3 \text{ and a.e. in } Q_T,\\ &(v_{a_n},w_{a_n}) \rightharpoonup (v,w) \text{ weakly in } [L^2(0,T;H^1(\Omega))]^2\end{aligned}$$

as $a_n \to 0$.

Proof. The estimates (16), (26) and (27) together with Corollaries 3.6 and 3.8 and Remark 3.1 imply the existence of a triple $(u,v,w) \in [L^\infty(Q_T)]^3$ satisfying (14) and such that $(v,w) \in [L^2(0,T;H^1(\Omega))]^2$ and $(v_t,w_t) \in [L^2(0,T;(H^1(\Omega))')]^2$, and a subsequence $\{(u_{a_n},v_{a_n},w_{a_n})\}_{a_n>0}$ such that

$$(u_{a_n},v_{a_n},w_{a_n}) \to (u,v,w) \quad \text{in every } L^p(Q_T),$$

$p \in [1,\infty)$, as $a_n \to 0$. □

Proof of Theorem 1.1. Repeatedly using the Lebesgue Dominated Convergence Theorem and Corollary 3.9 allow us to pass to the limit $a = a_n \to 0$ in the weak formulation (20), (21) and (22) to obtain (11), (12) and (13). The regularity result that $(v,w) \in [W_p^{2,1}(\Omega\times(\delta,T))]^2$ for all $\delta \in (0,T)$ follows from [7] (Lemma 3.4 on p. 206).

The uniqueness of the solution can be proved in a classical way by testing the equations for $U = u_1 - u_2$, $V = v_1 - v_2$ and $W = w_1 - w_2$ by U, V, and W, respectively, where (u_1,v_1,w_1) and (u_2,v_2,w_2) are two solutions of Problem $(\mathscr{P})$ such that $0 \le u_1,u_2,v_1,v_2 \le C_\varepsilon$ and $0 \le w_1,w_2 \le 1$. The uniform bounds (14), Lipschitz continuity of $\phi(z) = p(z)z$ and the Young inequality allow us to find a positive constant $C(\varepsilon)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (U^2+V^2+W^2) \le C(\varepsilon)\int_\Omega (U^2+V^2+W^2). \tag{40}$$

Indeed, from the equation for U we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega U^2 = \int_\Omega \Big\{&U^2 - (u_1+u_2)U^2 - v_1U^2 - u_2UV + sw_1U^2 + su_2UW\\ &- \frac{1}{\varepsilon}(\phi(u_1+v_1) - \phi(u_2+v_2))U + \frac{1}{\varepsilon}UV\Big\},\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega U^2 \le \int_\Omega (1+s)U^2 + C_\varepsilon\int_\Omega |UV| + sC_\varepsilon\int_\Omega |UW|&\\ + \frac{C_{Lip}}{\varepsilon}\int_\Omega |(U+V)U| + \frac{1}{\varepsilon}\int_\Omega |UV|.&\end{aligned}$$

The Young inequality then gives the first part of (40), namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega U^2 \le C(\varepsilon)\int_\Omega (U^2+V^2+W^2).$$

Analogously we obtain the estimates for V and W. The uniqueness then follows form the Gronwall inequality applied to (40). $\square$

4 Concluding remarks

In this paper we have proposed a reaction-diffusion/ODE model describing the interaction of farmers and hunter-gatherers in the Neolithic transition in Europe and addressed the fundamental question of existence and uniqueness of the solution of the system. This model is a combination of a Lotka-Volterra structure between the farmers and hunter-gatherers and the superimposed interaction between sedentary and migrating farmers. A key feature of this model is that the sedentary and migrating farmers convert to each other depending on the total density of farmers. Intuitively speaking, if the total density of farmers is relatively large, the sedentary farmers tend to actively convert to the migrating ones because of overcrowding, while if the total density of farmers is relatively small, the situation is reversed. As explained in the assumption (A2), this mechanism reminds the so-called population pressure effect in the farming population.

By passing to the limit in (5) as $\varepsilon \to 0$, we formally obtain

$$p(F)F_1 = (1-p(F))F_2,$$

that is,

$$F_2 = p(F)F \tag{41}$$

where $F = F_1 + F_2$. On the other hand, adding the first two equations in (5) together gives

$$\begin{aligned} F_t &= \Delta F_2 + (1-F)F + sFH, \\ H_t &= \Delta H + b(1-H)H - gFH. \end{aligned} \tag{42}$$

Therefore, in the limit $\varepsilon \to 0$, by substituting (41) in (42) we obtain

$$\begin{aligned} F_t &= \Delta(p(F)F) + (1-F)F + sFH, \\ H_t &= \Delta H + b(1-H)H - gFH. \end{aligned} \tag{43}$$

The first equation in (43) can be rewritten as

$$F_t = \mathrm{div}(D(F)\nabla F) + (1-F)F + sFH, \tag{44}$$

where $D(F) = p'(F)F + p(F)$. Since $D(F) = 0$ whenever $F = 0$, then the equation (44) is a degenerate, nonlinear diffusion equation. The diffusion $D(F)$ for $p(F) = p_m(F;F_c)$ with $m = 2$ and $F_c = 1$, where $p_m(F;F_c)$ is defined by (4), is shown in Fig. 5. The difference between the standard reaction-diffusion (F,H)-system with linear diffusion stated in (1) and the model (43) is obvious. Of course, if p is a constant function, then (43) coincides with (1).

The main purpose of this paper was to give a comprehensive introduction into modelling the Neolithic transition from hunting-gathering to farming by using fundamental mathematical framework.

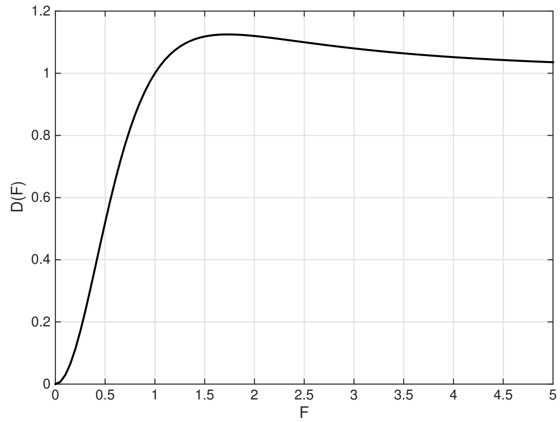

Figure 5: Functional form of $D(F) = p'(F)F + p(F)$ in (44) for $p(F) = p_m(F;F_c)$ defined by (4) with $m = 2$ and $F_c = 1$.

We do not pursue the study of the large time behaviour of the solution of (5)-(7) as well as the rigorous derivation of (44) from (5) as $\varepsilon \to 0$ in the present paper. These will be reported in forthcoming works.

Finally, we remark that the proposed model with a possibly modified interspecies dynamics can be applied to a variety of other biological and socio-economical migrations. For example, one can imagine a population of a predator (e.g., wolves) that follow the same "migration rules" as imposed by the assumptions (A1)-(A3). In particular, a predator occupying a territory with the sufficient food resources for its survival, stays in that territory. On the other hand, if the population of the predator grows to a certain size in this territory, some individuals leave the place and search actively for another uninhabited place or a place with low density of the predator. A prey can migrate freely and randomly. Another example may include a population that migrates into regions occupied by former residents. The migrants may decide to stay in a region or move according to the total population of migrants since, for example, a high density of migrants may mean less work possibilities for them. On the other hand, migrants can also decide to stay or move according to a "local rule", e.g., language spoken, so in that case they would prefer to move from a place with low to high density of people speaking the same language. In this scenario, a decreasing function p in the model (5) has to be assumed. Of course, no intense conversion of the former residents into migrants should be expected even though it cannot be completely excluded and the diffusion of the former residents should be small or even neglected from the modelling.

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