Singular limit for a reaction-diffusion-ODE system in a neolithic transition model

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Abstract

A reaction-diffusion-ODE model for the Neolithic spread of farmers in Europe has been recently proposed in [7]. In this model, farmers are assumed to be divided into two subpopulations according to a mobility rule, namely, into sedentary and migrating farming populations. The conversion between the farming subpopulations depends on the total density of farmers and it is superimposed on the classical Lotka-Volterra competition model, so that it is described by a three-component reaction-diffusion-ODE system. In this article we consider a singular limit problem when the conversion rate tends to infinity and prove under appropriate conditions that solutions of the three component system converge to solutions of a two-component system with a linear diffusion and nonlinear degenerate diffusion.

Keywords: reaction-diffusion system; Lotka-Volterra; singular limit problem; nonlinear degenerate diffusion

1 Introduction

Since the pioneering works by Fisher [9] and Kolmogorov-Petrovsky-Piskunov [10], reaction-diffusion equations are widely used for modelling propagation phenomena in the field of mathematical biology, specifically, population dynamics and population genetics. Besides the Fisher-KPP equation we can find many interesting model equations related to the propagation phenomena (see [13]). The recent works [6] and [7] treat new models for the propagation of the neolithic transition and numerically exhibit interesting transient spatial patterns in addition to mathematical studies.

In this article we deal with a model for the Neolithic transition which was developed in [7]. The spread of farmers into regions occupied by hunter-gatherers began during the neolithic transition period. Archeological evidence indicates that the expanding velocity of farmers is roughly constant all over Europe. In order to make this phenomenon clear, several population models have been proposed. One of the basic macroscopic models is a demic diffusion model described by the Fisher-KPP equation. As for a cultural diffusion model describing the expansion of farmers into the regions occupied by hunter-gatherers, Ammerman and Cavalli-Sforza ([1]) proposed a two component reaction-diffusion system for farmers and hunter-gatherers, which is a natural extension of the Fisher-KPP equation. In addition, there is a demic-cultural diffusion model, which is a mix of the two models above. It has the form of a three component system for the original farmer population, the migrating farmers and the hunter-gatherers ([2]). In those models it is assumed that the dispersal of farmers is random movement. However, one observes that the farmer populations are basically sedentary and that if the density becomes higher, they are forced to disperse. This indicates that the dispersal of farmers is not purely random.

Here we make the following assumptions:

(A1) farmers are basically sedentary,

(A2) if the density of sedentary farmers becomes higher, some of them become migratory and disperse randomly because of the population pressure,

(A3) if the environmental conditions become better for farmers, the migratory farmers stay in their region.

We note that the sedentary and the migratory farmers are not different from a genetic point of view. They change their movement according to the environmental conditions. This leads us to introduce two types of farmers, namely, the sedentary farmers and the migratory ones which convert into each other with some probability.

We make use of a reaction-diffusion-ODE model, proposed in [7], describing the spatio-temporal evolution of sedentary and migratory farmers and hunter–gatherers in the Neolithic transition. From the point of view of ecology, the model stems from the fact that a lifestyle of agriculture and settlement can support a much larger population density than hunting and gathering. Therefore, it is assumed in our modelling framework that farmers preferentially lived a sedentary lifestyle which could convert to a migratory one if the population of farmers grows over some critical densities. To the best of our knowledge, this assumption was not considered elsewhere except for [7] and [6].

We first introduce the model of sedentary and migrating farmers. Let F_1 and F_2 be the densities of sedentary and migrating farmers. The spatial and temporal evolution of farmers is modelled by the following system of equations

$$\begin{cases} F_{1,t} = r_1 \left(1 - \dfrac{F_1 + F_2}{K_{F_1}}\right) F_1 - \dfrac{1}{\varepsilon}\left(P(F)F_1 - (1-P(F))F_2\right), \\ F_{2,t} = d_{F_2}\Delta F_2 + r_2 \left(1 - \dfrac{F_1 + F_2}{K_{F_2}}\right) F_2 + \dfrac{1}{\varepsilon}\left(P(F)F_1 - (1-P(F))F_2\right), \end{cases} \tag{1}$$

where $F_{j,t} = \partial_t F_j$. In (1), r_1 and r_2 are the intrinsic growth rates of F_1 and F_2, K_{F_1} and K_{F_2} are the carrying capacities of F_1 and F_2, d_{F_2} is the rate of random dispersion of the migrating population of farmers F_2 and $F = F_1 + F_2$. The conversion between F_1 and F_2 is given by a probability density function P, which depends on the total population of farmers F, $P = P(F)$, and where $P(F)F_1$ is the conversion rate from F_1 to F_2 and $(1-P(F))F_2$ is the conversion rate from F_2 to F_1. The probability P satisfies

$$
\begin{aligned}
&(i) && P(0) = 0,\\
&(ii) && P'(F) > 0 \ \text{ for } \ F > 0,\\
&(iii) && \lim_{F\to\infty} P(F) = 1.
\end{aligned}
$$

As a specific example, we may consider

$$P(F) = \frac{F}{F + F_c}$$

where F_c is a positive constant such that $P(F_c) = 1/2$. The conversion mechanism can be best seen by considering the "extreme" situations, namely, when $F \gg F_c$ and $F \ll F_c$. In the first case when $F \gg F_c$, the probability $P(F)$ is high, say $P(F) \approx 1$, and we have that $P(F)F_1 - (1-P(F))F_2 \approx F_1$. This term in (1) implies that the sedentary farmers F_1 convert actively to the migrating farmers F_2. In the case when $F \ll F_c$ we have $P(F) \approx 0$ and so $P(F)F_1 - (1-P(F))F_2 \approx -F_2$. This term in (1) implies that the migrating farmers F_2 convert to the sedentary farmers F_1. The extent of conversion is given by the constant $1/\varepsilon$.

By taking the population of hunter-gatherers into account, we can extend the system (1) to the full farmer–hunter system

$$
\begin{cases}
F_{1,t} = r_1\left(1 - \dfrac{F_1+F_2}{K_{F_1}}\right)F_1 - \dfrac{1}{\varepsilon}\left(P(F)F_1 - (1-P(F))F_2\right) + e_{F_1}F_1H,\\
F_{2,t} = d_{F_2}\Delta F_2 + r_2\left(1 - \dfrac{F_1+F_2}{K_{F_2}}\right)F_2 + \dfrac{1}{\varepsilon}\left(P(F)F_1 - (1-P(F))F_2\right) + e_{F_2}F_2H,\\
H_t = d_H\Delta H + r_H\left(1 - \dfrac{H}{K_H}\right)H - e_{F_1}F_1H - e_{F_2}F_2H,
\end{cases}
\tag{2}
$$

where d_H, r_H and K_H are, respectively, the rate of dispersion, the intrinsic growth rate and the carrying capacity of H, and e_{F_1} and e_{F_2} are the conversion rates from H to F_1 and F_2. All parameters in (2) are positive constants.

Since the timescale of the conversion between F_1 and F_2 is fast, compared with the growth of F_1 and F_2 and the conversion from H to F_1 and F_2, we assume that ε is a small parameter in contrast to the other parameters in (2). Moreover, the sedentary and migrating farmers F_1 and F_2 are not genetically distinguished in the total population of farmers $F = F_1 + F_2$. Hence, it leads us to the natural question whether the system (2) for (F_1, F_2, H) can be reduced to a system for (F, H). In fact, adding the equations of F_1 and F_2, we obtain an equation for F. However, it is not a closed system for (F, H). Taking $\varepsilon \to 0$, we formally obtain the relation $P(F)F - F_2 = 0$ in the limit. Then the system becomes a closed system. We note that the limit system is no longer a simple reaction-diffusion system for (F, H) but a degenerate, nonlinear diffusion system which includes the conversion rate $P(F)$ in the diffusion term. This kind of limiting procedure is called a fast reaction limit and has been extensively used in literature ([5], [8], [12] et al.) in order to prove the validity of the formal procedure. However, by the conditions for P, which are natural in our model, the previous results are not applicable to our case (see [12]). The purpose of this paper is to apply a fast reaction limit method and prove that the limit system approximates the original system in a suitable way.

The organisation of this paper is as follows. In Section 2 we state the main theorems in addition to the notations and definitions. In Section 3 we introduce the Fréchet-Kolmogorov compactness theorem, which plays a key role to obtain the main results. In Section 4 we provide a priori L^∞-estimates which are uniform in ε and μ. While we apply maximum principle arguments to find an upper bound for the function F_1, we estimate L^p-norms of F_2 to finally obtain a L^∞-bound. In Section 5 we use the a priori estimates to prove the relative compactness of sequences of solutions. Finally, in Section 6 we complete the proof of the main theorems.

2 Preliminaries and main results

We assume

$$K := K_{F_1} = K_{F_2}, \qquad e_f := e_{F_1} = e_{F_2}, \qquad r := r_1 = r_2,$$

and set

$$a := e_f K_H / r, \qquad b := e_f K / r, \qquad \varepsilon' = \varepsilon r, \qquad d := d_{F_2}/d_H, \quad r' = r_H/r.$$

After making the transformations in (2) as

$$x' := \sqrt{r/d_H}, \quad t' := rt, \quad u := F_1/K, \quad v := F_2/K, \quad w := H/K_H,$$

and dropping the primes, we obtain a system of equations for

$$(\mathscr{P}_\varepsilon) \begin{cases} u_t = (1-u-v)u + auw - \dfrac{1}{\varepsilon}(\varphi(u+v)-v) & \text{in } Q_T, \\ v_t = d\Delta v + (1-u-v)v + avw + \dfrac{1}{\varepsilon}(\varphi(u+v)-v) & \text{in } Q_T, \\ w_t = \Delta w + r(1-w)w - b(u+v)w & \text{in } Q_T, \\ \partial_\nu v = \partial_\nu w = 0 & \text{on } \Gamma_T, \\ (u(\cdot,0), v(\cdot,0), w(\cdot,0)) = (u_0, v_0, w_0) & x \in \Omega, \end{cases}$$

where we put $\varphi(s) := sP(Ks)$. We suppose that Ω is an open, bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary (e.g., $\partial\Omega \in C^2$), $Q_T = \Omega \times (0,T)$ and $\Gamma_T = \partial\Omega \times (0,T)$ for an arbitrary $T > 0$. The coefficients a, b, c, d, r and ε are positive constants. By $W^{2,1}(\Omega)$ we denote the space of all functions f such that $f, \nabla f, \Delta f \in L^1(\Omega)$.

We assume

$$(H_\varphi) \quad \begin{cases} \varphi \in C^2(\mathbb{R}_+), \ \ \varphi(0) = \varphi'(0) = 0, \\ 0 < \varphi(s) < s\varphi'(s) \ \text{ and } \ 0 < \varphi'(s) < 1 \ \text{ for } \ s > 0, \ \ \lim_{s\to\infty}(s - \varphi(s)) = \tilde{C}, \\ \varphi''(s) \geq 0 \ \text{ for } \ s \in [0, 2(1+a)), \end{cases}$$

for a positive number $\tilde{C}$. We remark that the last assumption is necessary for the proof of the existence of a unique solution of Problem $(\mathscr{P})$ which will be given below.

The next lemma shows that the probability density $P(s)$ follows from $\varphi(s)$ satisfying (H_φ).

Lemma 2.1. *Under the assumptions of* (H_φ) *the probability density* $P(s) := \varphi(s/K)/(s/K)$ $(s > 0)$ *satisfies*

(i) $\lim_{s\to 0} P(s) = 0$,

(ii) $P'(s) > 0$ for $s > 0$,

(iii) $\lim_{s\to\infty} P(s) = 1$.

Proof. The assertion of (i) immediately follows from the definition of $P(s)$ and $\varphi'(0) = 0$. We compute

$$P'(s) = \frac{\varphi'(s/K)}{s} - \frac{K\varphi(s/K)}{s^2} = \frac{K[(s/K)\varphi'(s/K) - \varphi(s/K)]}{s^2} > 0.$$

Finally the assumptions of (H_φ) imply that $s - \varphi(s)$ is monotone increasing and $s - \varphi(s) < \tilde{C}$, so

$$s - \tilde{C} < \varphi(s) < s. \tag{3}$$

Thus

$$\frac{s/K - \tilde{C}/K}{s/K} < \frac{\varphi(s/K)}{s/K} = P(s) < 1,$$

which yields $\lim_{s\to\infty} P(s) = 1$. □

Lemma 2.2. *With* $\tilde{P}(s) := s/(s+F_c)$*, the function* $\tilde{\varphi}(s) := s\tilde{P}(Ks)$ *satisfies* (H_φ) *for* $\tilde{C} = F_c/K$.

Proof. We first write

$$\tilde{\varphi}(s) = \frac{s^2}{s + F_c/K} = \frac{s^2}{s+\tilde{C}} = s\left(1 - \frac{\tilde{C}}{s+\tilde{C}}\right).$$

Then we easily see

$$\tilde{\varphi} \in C^2(\mathbb{R}_+), \quad \tilde{\varphi}(0) = \tilde{\varphi}'(0) = 0$$

hold. On the other hand

$$0 < \tilde{\varphi}'(s) = \frac{s(s+\tilde{C})}{(s+\tilde{C})^2} = 1 - \frac{\tilde{C}^2}{(s+\tilde{C})^2} < 1 \quad \text{for} \quad s > 0,$$

and

$$s\tilde{\varphi}'(s) - \tilde{\varphi}(s) = \frac{s^2\tilde{C}}{(s+\tilde{C})^2} > 0 \quad \text{for} \quad s > 0.$$

Finally,

$$\lim_{s\to\infty}(s - \tilde{\varphi}(s)) = \lim_{s\to\infty}\frac{s\tilde{C}}{s+\tilde{C}} = \tilde{C} \text{ and } \varphi''(s) = \frac{2\tilde{C}^2}{(s+\tilde{C})^3},$$

which completes the proof. □

By the monotonicity of φ we can define the inverse φ^{-1} on $[0,\infty)$ and it follows from (3) that

$$s < \varphi^{-1}(s) < s + \tilde{C} \quad \text{for } s \geq 0. \tag{4}$$

Indeed, the second inequality in (3) yields the first inequality in (4) and the first inequality in (3) is equivalent to $s < \varphi(s+\tilde{C})$ for all $s \geq -\tilde{C}$ which leads to the second inequality in (4). Moreover, we define $\alpha(s) := s - \varphi(s)$ and assume $\tilde{C} > 1+a$. Then $0 < \alpha'(s) < 1$ for $s > 0$ and α is a strictly increasing function on $\mathbb{R}$ with values in $[0,\tilde{C})$. The inverse function α^{-1} is strictly increasing on $[0,\tilde{C})$, and so on $[0,1+a]$. As a consequence of the monotonicity of α^{-1} on $[0,\tilde{C})$ and in view of (H_φ), we have

$$s \le \alpha^{-1}(s) \le C_2 s \quad \text{for } 0 \le s \le 1+a, \quad C_2 := \frac{\alpha^{-1}(1+a)}{1+a}, \tag{5}$$

where the second inequality follows from the convexity of the graph of α^{-1} in $[0,1+a]$, indeed $\alpha''(s) = -\varphi''(s) \le 0$ in $0 \le s \le 1+a$. These two inverse functions play important roles in the proof of the main result.

Next, we assume that the initial functions $u_0, v_0, w_0 \in W^{2,1}(\Omega) \cap C(\overline{\Omega})$ satisfy

$$\begin{aligned} &(H_0)(i) \qquad u_0 \ge 0, \quad v_0 \ge 0, \quad 0 \le u_0+v_0 \le 1+a \quad \text{and} \quad 0 \le w_0 \le 1 \quad \text{in} \quad \overline{\Omega}, \\ &(H_0)(ii) \qquad \varphi(u_0+v_0) = v_0 \quad \text{in} \quad \overline{\Omega}. \end{aligned}$$

We are now in a position to state our main theorem on the singular limit of Problem $(\mathscr{P}_\varepsilon)$ as ε converges to 0.

Theorem 2.3. *Assume (H_φ) and $\tilde{C} > 1+a$. Let $T > 0$. Let $\{u^\varepsilon\}_{\varepsilon>0}$, $\{v^\varepsilon\}_{\varepsilon>0}$ and $\{w^\varepsilon\}_{\varepsilon>0}$ be solutions to $(\mathscr{P}_\varepsilon)$ satisfying $(H_0)(i)-(ii)$ and let $p \in [1,\infty)$ be arbitrary. Then there exist subsequences $\{u^{\varepsilon_k}\}_{\varepsilon_k>0}$, $\{v^{\varepsilon_k}\}_{\varepsilon_k>0}$ and $\{w^{\varepsilon_k}\}_{\varepsilon_k>0}$ of the sequences $\{u^\varepsilon\}_{\varepsilon>0}$, $\{v^\varepsilon\}_{\varepsilon>0}$ and $\{w^\varepsilon\}_{\varepsilon>0}$, functions $u \in L^\infty(Q_T)$, $v \in L^\infty(Q_T)$ and $w \in L^\infty(Q_T)$ such that*

$$u^{\varepsilon_k} \to u, \quad v^{\varepsilon_k} \to v, \quad w^{\varepsilon_k} \to w \quad \textit{strongly in } L^p(Q_T) \textit{ and a.e. in } Q_T \tag{6}$$

as $\varepsilon_k \to 0$.

Let us denote

$$z^{\varepsilon_k} = u^{\varepsilon_k} + v^{\varepsilon_k} \quad \text{and} \quad z = u+v,$$

where u and v are the limit functions in Theorem 2.3. We deduce from Theorem 2.3 that for $p \in [1,\infty)$

$$z^{\varepsilon_k} \to z \in L^\infty(Q_T) \quad \text{strongly in } L^p(Q_T) \text{ and a.e. in } Q_T \tag{7}$$

as $\varepsilon_k \to 0$. Then we can assert that (z,w) is the unique weak solution of Problem

$$(\mathscr{P}) \begin{cases} z_t = d\Delta\varphi(z) + (1-z)z + azw & \text{in } Q_T, \\ w_t = \Delta w + r(1-w)w - bzw & \text{in } Q_T, \\ \partial_\nu \varphi(z) = \partial_\nu w = 0 & \text{on } \Gamma_T, \\ (z(\cdot,0), w(\cdot,0)) = (z_0, w_0), & x \in \Omega, \end{cases}$$

where $z_0 = u_0 + v_0$ is such that $0 \le z_0 \le 1+a$. Following [3], we define a weak solution of Problem $(\mathscr{P})$ as $z, w \in C([0,T]; L^1(\Omega)) \cap L^\infty(Q_T)$ and

$$-\int_\Omega z_0 \xi(0)\,\mathrm{d}x = \iint_{Q_T} (d\varphi(z)\Delta\xi + ((1-z)z + azw)\xi + z\xi_t)\,\mathrm{d}x\,\mathrm{d}t, \tag{8}$$

$$-\int_\Omega w_0 \xi(0)\,\mathrm{d}x = \iint_{Q_T} (w\Delta\xi + (r(1-w)w - bzw)\xi + w\xi_t)\,\mathrm{d}x\,\mathrm{d}t \tag{9}$$

hold for all test functions $\xi \in C^{2,1}(\overline{Q}_T)$ such that $\xi(x,T)=0$ in Ω and $\partial_\nu \xi = 0$ on $\partial\Omega \times [0,T]$.

We remark that the existence and uniqueness of the weak solution of Problem $(\mathscr{P})$ can be established by modifying the arguments in [3] and [6]. In fact, the well-posedness of Problem $(\mathscr{P})$ and the uniqueness for a weak solution $z,w \in C(\overline{Q_T})$ as well as the large time behaviour of solutions of Problem $(\mathscr{P})$ has been studied in [6]. In particular, it is proved in [6] that any solution (z,w) of Problem $(\mathscr{P})$ satisfies

$$0 \le z \le 1+a \quad \text{and} \quad 0 \le w \le 1 \tag{10}$$

in $\overline{\Omega} \times [0,\infty)$. Moreover, the solution orbits converge to a steady state uniformly in $C(\overline{\Omega})^2$ as $t \to \infty$ and, depending on the coefficient b, the convergence has either exponential rate if $b \neq 1$ or algebraic rate if $b=1$ in the L^p topology for each $p \in [1,\infty)$.

We remark that the limit functions u and v are given by $v = \varphi(z)$ and $u = z - \varphi(z)$ and that the equation for z is parabolic degenerate since $\varphi'(0)=0$ while the equation for w is uniformly parabolic.

In the sequel we obtain the next theorem.

Theorem 2.4. *Assume the same assumptions as in Theorem 2.3. The pair of functions $(z,w) = (u+v,w)$ given by the limit functions (u,v,w) in Theorem 2.3 coincides with the unique weak solution of Problem $(\mathscr{P})$ with $z_0 = u_0+v_0 \in C(\overline{\Omega})$ and $w_0 \in C(\overline{\Omega})$.*

As in [7], we consider the regularised problem

$$(\mathscr{P}_{\varepsilon,\mu}) \begin{cases} u_t = \mu\Delta u + (1-u-v)u + auw - \dfrac{1}{\varepsilon}(\varphi(u+v)-v) & \text{in } Q_T, \\ v_t = d\Delta v + (1-u-v)v + avw + \dfrac{1}{\varepsilon}(\varphi(u+v)-v) & \text{in } Q_T, \\ w_t = \Delta w + r(1-w)w - b(u+v)w \quad \text{in } Q_T, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0 \quad \text{on } \Gamma_T, \\ (u(\cdot,0),v(\cdot,0),w(\cdot,0)) = (u_0,v_0,w_0) \quad x \in \Omega, \end{cases}$$

where $0<\mu<M$ for some positive constant M. Problem $(\mathscr{P}_{\varepsilon,\mu})$ admits a unique nonnegative classical solution $(u^\varepsilon_\mu, v^\varepsilon_\mu, w^\varepsilon_\mu) \in [C^{2,1}(\overline{\Omega}\times(0,T]) \cap C(\overline{\Omega}\times[0,T])]^3$. Moreover, it holds that

$$0 \le u^\varepsilon_\mu, v^\varepsilon_\mu \quad \text{and} \quad 0 \le w^\varepsilon_\mu \le 1 \tag{11}$$

in $\overline{Q}_T$.

It is proved in [7] that the solution $(u^\varepsilon_\mu, v^\varepsilon_\mu, w^\varepsilon_\mu)$ of Problem $(\mathscr{P}_{\varepsilon,\mu})$ converges to the unique solution $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ of Problem $(\mathscr{P}_\varepsilon)$ as $\mu \to 0$ for $\varepsilon > 0$, where

$$\begin{aligned} &u^\varepsilon \in C^{0,1}([0,T];L^\infty(\Omega)), \\ &v^\varepsilon, w^\varepsilon \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),\ v^\varepsilon_t, w^\varepsilon_t \in L^2(0,T;(H^1(\Omega))'), \end{aligned} \tag{12}$$

and

$$-\int_\Omega u_0\xi(0)\,\mathrm{d}x = \iint_{Q_T} \left(\left((1-u^\varepsilon-v^\varepsilon)u^\varepsilon + au^\varepsilon w^\varepsilon - \frac{1}{\varepsilon}(\varphi(u^\varepsilon+v^\varepsilon)-v^\varepsilon) \right)\xi + u^\varepsilon \xi_t \right) \mathrm{d}x\,\mathrm{d}t, \tag{13}$$

$$-\int_\Omega v_0\xi(0)\,\mathrm{d}x = \iint_{Q_T} \left(dv^\varepsilon\Delta\xi + \left((1-u^\varepsilon-v^\varepsilon)v^\varepsilon + av^\varepsilon w^\varepsilon + \frac{1}{\varepsilon}(\varphi(u^\varepsilon+v^\varepsilon)-v^\varepsilon) \right)\xi + v^\varepsilon \xi_t \right) \mathrm{d}x\,\mathrm{d}t, \tag{14}$$

$$-\int_\Omega w_0\xi(0)\,\mathrm{d}x = \iint_{Q_T} (w^\varepsilon\Delta\xi + (r(1-w^\varepsilon)w^\varepsilon - b(u^\varepsilon+v^\varepsilon)w^\varepsilon)\xi + w^\varepsilon\xi_t)\,\mathrm{d}x\,\mathrm{d}t, \tag{15}$$

for $\xi \in C^{2,1}(\overline{Q}_T)$ such that $\xi(x,T) = 0$ in Ω and $\partial_\nu \xi = 0$ on $\partial\Omega \times [0,T]$.

In a similar way we can obtain the singular limit of Problem $(\mathscr{P}_{\varepsilon,\mu})$ as $\varepsilon \to 0$ for $\mu > 0$, namely Problem $(\mathscr{P}_\mu)$,

$$(\mathscr{P}_\mu)\begin{cases} z_t = \mu\Delta(z-\varphi(z)) + d\Delta\varphi(z) + (1-z)z + azw & \text{in } Q_T, \\ w_t = \Delta w + r(1-w)w - bzw & \text{in } Q_T, \\ \partial_\nu \varphi(z) = \partial_\nu w = 0 & \text{on } \Gamma_T, \\ (z(\cdot,0), w(\cdot,0)) = (z_0, w_0), & x \in \Omega, \end{cases}$$

where $z_0 = u_0 + v_0$. One could also prove that the singular limit of Problem $(\mathscr{P}_\mu)$ as $\mu \to 0$ is given by Problem $(\mathscr{P})$. Consequently, we can complete the graph on Figure 1.

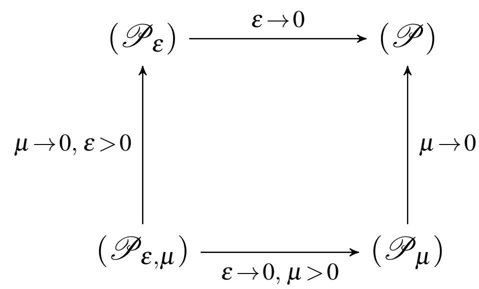

Figure 1: Schematic representation of the singular limit problems: A regularised problem $(\mathscr{P}_{\varepsilon,\mu})$ is obtained from Problem $(\mathscr{P}_\varepsilon)$ by adding a diffusion term into the equation for u. The singular limit of Problem $(\mathscr{P}_{\varepsilon,\mu})$ as $\mu \to 0$ for $\varepsilon > 0$ is shown in [7]. While the a priori estimates depend on ε in [7], the uniform estimates in ε and μ, which we obtain in this paper, allow us to prove that the singular limit of Problem $(\mathscr{P}_\varepsilon)$ as $\varepsilon \to 0$ is given by Problem $(\mathscr{P})$. Similarly, we can show the singular limit of Problem $(\mathscr{P}_{\varepsilon,\mu})$ as $\varepsilon \to 0$ for $\mu > 0$ and the singular limit of Problem $(\mathscr{P}_\mu)$ as $\mu \to 0$.

We therefore treat the regularised problem $(\mathscr{P}_{\varepsilon,\mu})$ and prove some uniform estimates in ε, μ and $p \geq 2$ in the later sections.

3 Compactness theorem

The main tool that is used in this present paper is the following Fréchet-Kolmogorov compactness theorem, see [4], Theorem 4.26 on p. 111; the form below is taken from [5], Proposition 2.5.

Theorem 3.1 (Fréchet-Kolmogorov)**.** *Let $\mathscr{F}$ be a bounded subset of $L^p(Q_T)$ with $p \in [1,\infty)$. Assume that*

i) for any $\eta > 0$ and any subset $\omega \Subset Q_T$, there exists $\delta > 0$ ($\delta < dist(\omega, \partial Q_T)$) such that

$$\|f(x+\xi,t) - f(x,t)\|_{L^p(\omega)} + \|f(x,t+\tau) - f(x,t)\|_{L^p(\omega)} < \eta$$

for all ξ, τ and $f \in \mathscr{F}$ satisfying $|\xi| + |\tau| < \delta$.

ii) for any $\eta > 0$, there exists a subset $\omega \Subset Q_T$ such that

$$\|f\|_{L^p(Q_T \setminus \omega)} < \eta$$

for all $f \in \mathscr{F}$.

Then $\mathscr{F}$ is relatively compact in $L^p(Q_T)$.

To apply the Fréchet-Kolmogorov theorem, we consider two subsets of Ω, namely $\Omega_r = \{x \in \Omega \,|\, B(x,2r) \subset \Omega\}$ and $\Omega_r' = \cup_{x \in \Omega_r} B(x,r)$, where $B(x,r)$ denotes the ball in $\mathbb{R}^N$ with centre x and radius r. We have $\Omega_r \subset \Omega_r' \subset \Omega$. Moreover, we define a smooth function $\psi \in C_0^\infty(\Omega_r')$ such that

$$\psi = 1 \text{ in } \Omega_r, \quad 0 \le \psi \le 1 \text{ in } \Omega_r', \quad \psi = 0 \text{ in } \Omega \setminus \Omega_r'. \tag{16}$$

We refer to [8] for the precise construction of ψ. Let the sign function be defined by

$$\operatorname{sgn}(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -1 & \text{if } s < 0. \end{cases}$$

By sgn_η we denote a smooth nondecreasing approximation of the sign function such that $\operatorname{sgn}_\eta(s) \in [-1,1]$ for $s \in \mathbb{R}$ and sgn_η converges pointwise to sgn as $\eta \to 0$. For example, $\operatorname{sgn}_\eta(s) = s/\sqrt{s^2 + \eta^2}$.

4 Uniform L^∞-estimates

In this section we show that the solution $(u_\mu^\varepsilon, v_\mu^\varepsilon, w_\mu^\varepsilon)$ of $(\mathscr{P}_{\varepsilon,\mu})$ is such that u_μ^ε and v_μ^ε are bounded in $L^\infty(Q_T)$ uniformly in ε and μ. First we use a comparison principle to show the uniform boundedness of u_μ^ε.

Lemma 4.1. *It holds that*

$$0 \le u_\mu^\varepsilon < \tilde{C} \tag{17}$$

in $\overline{\Omega} \times [0,\infty)$, where $\tilde{C}$ is given by (H_φ) and satisfies $\tilde{C} > 1 + a$.

Proof. Let us define

$$\mathscr{L}_u(s) = s_t - \mu \Delta s - (1 - s - v_\mu^\varepsilon)s - asw_\mu^\varepsilon + \frac{1}{\varepsilon}(\varphi(s + v_\mu^\varepsilon) - v_\mu^\varepsilon).$$

We deduce from (H_φ), namely from the lower bound $\varphi(s) \ge s - \tilde{C}$ and (11) that

$$\begin{aligned} \mathscr{L}_u(\tilde{C}) &= \tilde{C} v_\mu^\varepsilon - (1 + aw_\mu^\varepsilon - \tilde{C})\tilde{C} + \frac{1}{\varepsilon}(\varphi(\tilde{C} + v_\mu^\varepsilon) - v_\mu^\varepsilon) \\ &\ge -(1 + a - \tilde{C})\tilde{C} + \frac{1}{\varepsilon}(\tilde{C} + v_\mu^\varepsilon - \tilde{C} - v_\mu^\varepsilon) \\ &= (\tilde{C} - (1+a))\tilde{C}. \end{aligned}$$

Since $\tilde{C} > 1 + a$ we see that $\mathscr{L}_u(\tilde{C}) > 0$. Hence, in view of the hypothesis (H_0), namely, $0 \le u_0 \le 1 + a$, it follows from the standard comparison principle that $u_\mu^\varepsilon < \tilde{C}$ for all $(x,t) \in \overline{\Omega} \times [0,\infty)$. □

For later use we define $C_u = \max_{(x,t)\in\overline{\Omega}\times[0,\infty)} u^\varepsilon_\mu(x,t)$. Then, it follows from (17) that

$$0 \le u^\varepsilon_\mu \le C_u < \tilde{C} \tag{18}$$

in $\overline{\Omega}\times[0,\infty)$.

However, we cannot obtain a similar upper bound for the function v^ε_μ. Nevertheless, we can still obtain a uniform L^∞ estimate for v^ε_μ, which we do in Corollary 4.5. First we prove some auxiliary lemmas. We remark that in view of (H_φ), $\alpha(s) = 1-\varphi(s)$ is strictly increasing from zero to $\tilde{C}$ and α^{-1} is well defined on the interval $[0,\tilde{C})$. Moreover, $\alpha^{-1}(0) = \varphi^{-1}(0) = 0$.

Lemma 4.2. *Let $p \in \mathbb{N}$, $p \ge 2$, and $Z = U+V$, where $0 < U < C$ and $V > 0$. Then,*

$$(\varphi(Z)-V)\left((\alpha^{-1}(U))^{p-1} - (\varphi^{-1}(V))^{p-1}\right) \ge 0. \tag{19}$$

Proof. Since $\varphi(Z)-V = U-\alpha(Z)$, we have that

$$\begin{aligned}
&(\varphi(Z)-V)\left((\alpha^{-1}(U))^{p-1} - (\varphi^{-1}(V))^{p-1}\right)\\
&\quad= (\varphi(Z)-V)\left((\alpha^{-1}(U))^{p-1} - Z^{p-1}\right) + (\varphi(Z)-V)\left(Z^{p-1} - (\varphi^{-1}(V))^{p-1}\right)\\
&\quad= (U-\alpha(Z))\left((\alpha^{-1}(U))^{p-1} - Z^{p-1}\right) + (\varphi(Z)-V)\left(Z^{p-1} - (\varphi^{-1}(V))^{p-1}\right)\\
&\quad= \left(\alpha(\alpha^{-1}(U)) - \alpha(Z)\right)\left((\alpha^{-1}(U))^{p-1} - Z^{p-1}\right)\\
&\qquad+ \left(\varphi(Z) - \varphi(\varphi^{-1}(V))\right)\left(Z^{p-1} - (\varphi^{-1}(V))^{p-1}\right)\\
&\quad\ge 0,
\end{aligned}$$

where the last inequality follows from the monotonicity of functions α, φ and $s \mapsto s^{p-1}$ for $p > 1$. □

Next for $p \in [2,\infty)$ we set

$$\Phi_\alpha(r) = \int_0^r (\alpha^{-1}(s))^{p-1}\,\mathrm{d}s \quad \text{and} \quad \Phi_\varphi(r) = \int_0^r (\varphi^{-1}(s))^{p-1}\,\mathrm{d}s. \tag{20}$$

Since α^{-1} and φ^{-1} are nonnegative functions, it holds that $\Phi_\alpha(r) \ge 0$ and $\Phi_\varphi(r) \ge 0$ for each r from the corresponding domains of definition.

Lemma 4.3. *Suppose that (H_0) and (H_φ) are satisfied. Let $p \in \mathbb{N}$, $p \ge 2$. Then, for $t > 0$*

$$\int_\Omega \Phi_\varphi(v^\varepsilon_\mu(x,t))\,\mathrm{d}x \ge \frac{1}{p}\int_\Omega (v^\varepsilon_\mu(x,t))^p\,\mathrm{d}x, \tag{21}$$

and there exists a constant $C_3 = C_3(p)$ such that

$$\int_\Omega \left(\Phi_\alpha(u_0) + \Phi_\varphi(v_0)\right)\mathrm{d}x \le C_3. \tag{22}$$

Proof. It follows from the lower bound in (4), namely, $\varphi^{-1}(s) \ge s$ for $s \ge 0$, that

$$\int_\Omega \Phi_\varphi(v^\varepsilon_\mu)\,\mathrm{d}x = \int_\Omega\int_0^{v^\varepsilon_\mu} (\varphi^{-1}(s))^{p-1}\,\mathrm{d}s\,\mathrm{d}x \ge \int_\Omega\int_0^{v^\varepsilon_\mu} s^{p-1}\,\mathrm{d}s\,\mathrm{d}x = \frac{1}{p}\int_\Omega (v^\varepsilon_\mu)^p\,\mathrm{d}x,$$

which completes the proof of (21). Next we prove (22). It follows from the upper bound in (4), namely, $\varphi^{-1}(s) \leq s + \tilde{C}$ for $s \geq 0$, that

$$\int_\Omega \Phi_\varphi(v_0)\,\mathrm{d}x = \int_\Omega \int_0^{v_0} (\varphi^{-1}(s))^{p-1}\,\mathrm{d}s\,\mathrm{d}x \leq \int_\Omega \int_0^{v_0} (s+\tilde{C})^{p-1}\,\mathrm{d}s\,\mathrm{d}x$$
$$= \int_\Omega \frac{1}{p}((v_0+\tilde{C})^p - \tilde{C}^p)\,\mathrm{d}x \leq \frac{((1+a)+\tilde{C})^p|\Omega|}{p},$$

where we used the uniform bound $0 \leq v_0 \leq 1+a$ from (H_0). Similarly, it follows from the upper bound in (5), namely, $\alpha^{-1}(s) \leq C_2 s$ for $0 \leq s \leq 1+a$, and the inequality $0 \leq u_0 \leq 1+a$ from (H_0) that

$$\int_\Omega \Phi_\alpha(u_0)\,\mathrm{d}x = \int_\Omega \int_0^{u_0} (\alpha^{-1}(s))^{p-1}\,\mathrm{d}s\,\mathrm{d}x \leq \int_\Omega \int_0^{u_0} (C_2 s)^{p-1}\,\mathrm{d}s\,\mathrm{d}x$$
$$= C_2^{p-1} \int_\Omega \frac{u_0^p}{p}\,\mathrm{d}x \leq \frac{C_2^{p-1}(1+a)^p|\Omega|}{p}.$$

Thus, setting

$$C_3 = \left(((1+a)+\tilde{C})^p + C_2^{p-1}(1+a)^p\right)\frac{|\Omega|}{p}, \tag{23}$$

we obtain (22). □

Lemma 4.4. *Suppose that (H_φ) and $(H_0)(i)$ are satisfied, and let $p \in \mathbb{N}$, $p \geq 2$. Then, there exists a constant $C_4 = C_4(p)$ such that for $t > 0$*

$$\int_0^t \int_\Omega (u_\mu^\varepsilon(\alpha^{-1}(u_\mu^\varepsilon))^{p-1} + v_\mu^\varepsilon(\varphi^{-1}(v_\mu^\varepsilon))^{p-1})\,\mathrm{d}x\,\mathrm{d}s \leq C_4 t + p\int_0^t \int_\Omega \Phi_\varphi(v_\mu^\varepsilon)\,\mathrm{d}x\,\mathrm{d}s. \tag{24}$$

Proof. It follows from the uniform bound (18) that $C_u = \max_{(x,t)\in\overline{\Omega}\times[0,\infty)} u_\mu^\varepsilon(x,t) < \tilde{C}$. Since α^{-1} is nondecreasing then

$$\int_0^t \int_\Omega u_\mu^\varepsilon(\alpha^{-1}(u_\mu^\varepsilon))^{p-1}\,\mathrm{d}x\,\mathrm{d}s \leq C_u(\alpha^{-1}(C_u))^{p-1}|\Omega|t = C_4 t, \tag{25}$$

where

$$C_4 = C_u(\alpha^{-1}(C_u))^{p-1}|\Omega|. \tag{26}$$

Next, we prove for $r \geq 0$ that

$$\frac{1}{p} r(\varphi^{-1}(r))^{p-1} \leq \int_0^r (\varphi^{-1}(s))^{p-1}\,\mathrm{d}s. \tag{27}$$

To this end we define

$$Q(r) = \int_0^r (\varphi^{-1}(s))^{p-1}\,\mathrm{d}s - \frac{1}{p} r(\varphi^{-1}(r))^{p-1}$$

and show that $Q(r) \geq 0$ for $r \geq 0$. We have that $Q(0) = 0$ and

$$Q'(r) = (\varphi^{-1}(r))^{p-2}\left(\varphi^{-1}(r) - \frac{1}{p}\varphi^{-1}(r) - \frac{p-1}{p} r\left(\varphi^{-1}\right)'(r)\right)$$
$$= \frac{p-1}{p}(\varphi^{-1}(r))^{p-2}\left(\varphi^{-1}(r) - r\left(\varphi^{-1}\right)'(r)\right).$$

Next we check that for all $r > 0$

$$r\left(\varphi^{-1}\right)'(r) \leq \varphi^{-1}(r). \tag{28}$$

Let $r = \varphi(s)$. Then, $s = \varphi^{-1}(r)$, $\left(\varphi^{-1}\right)'(r) = 1/\varphi'(s)$ and (28) is equivalent to

$$\frac{\varphi(s)}{\varphi'(s)} \leq s \Longleftrightarrow 1 \leq \frac{s\varphi'(s)}{\varphi(s)}$$

which, in view of (H_φ), is true for each $s > 0$. Thus, since $p \geq 2$ and $\varphi^{-1}(r) > 0$ for $r > 0$ we deduce that $Q'(r) \geq 0$ for $r > 0$. This with $Q(0) = 0$ implies that $Q(r) \geq 0$ for $r \geq 0$, which yields (27). In view of (20) we rewrite (27) as

$$r(\varphi^{-1}(r))^{p-1} \leq p\Phi_\varphi(r) \tag{29}$$

for all $r \geq 0$. We set $r = v_\mu^\varepsilon$ and integrate (29) on space and time to obtain that

$$\int_0^t \int_\Omega v_\mu^\varepsilon(\varphi^{-1}(v_\mu^\varepsilon))^{p-1}\,\mathrm{d}x\,\mathrm{d}s \leq p\int_0^t \int_\Omega \Phi_\varphi(v_\mu^\varepsilon)\,\mathrm{d}x\,\mathrm{d}s,$$

which together with (25) completes the proof of (24). □

Corollary 4.5. *Suppose that (H_φ) and $(H_0)(i)$ are satisfied. Then, there exists a constant $C_v = C_v(T) > 0$ independent of ε and μ such that*

$$0 \leq v_\mu^\varepsilon \leq C_v \tag{30}$$

in $\overline{\Omega} \times [0,T]$.

Proof. We remark that for all smooth enough functions $f = f(t)$, there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\alpha(f) = (\alpha^{-1}(f))^{p-1}\frac{\mathrm{d}f}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\Phi_\varphi(f) = (\varphi^{-1}(f))^{p-1}\frac{\mathrm{d}f}{\mathrm{d}t},$$

where the functions Φ_α and Φ_φ are defined in (20). Multiplying the equations for u_μ^ε and v_μ^ε in Problem $(\mathscr{P}_{\varepsilon,\mu})$ by $(\alpha^{-1}(u_\mu^\varepsilon))^{p-1}$ and $(\varphi^{-1}(v_\mu^\varepsilon))^{p-1}$ for $p \in [2,\infty)$, respectively, adding the results and integrating in space and time yield for $t > 0$

$$\begin{aligned}
&\int_\Omega (\Phi_\alpha(u_\mu^\varepsilon(t)) + \Phi_\varphi(v_\mu^\varepsilon(t)))\,\mathrm{d}x + (p-1)\mu\int_0^t\int_\Omega (\alpha^{-1}(u_\mu^\varepsilon))^{p-2}(\alpha^{-1})'(u_\mu^\varepsilon)\left|\nabla u_\mu^\varepsilon\right|^2\mathrm{d}x\,\mathrm{d}s \\
&\qquad + (p-1)d\int_0^t\int_\Omega (\varphi^{-1}(v_\mu^\varepsilon))^{p-2}(\varphi^{-1})'(v_\mu^\varepsilon)\left|\nabla v_\mu^\varepsilon\right|^2\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \frac{1}{\varepsilon}\int_0^t\int_\Omega \left(\varphi(z_\mu^\varepsilon) - v_\mu^\varepsilon\right)\left((\alpha^{-1}(u_\mu^\varepsilon))^{p-1} - (\varphi^{-1}(v_\mu^\varepsilon))^{p-1}\right)\mathrm{d}x\,\mathrm{d}s \\
&\leq \int_\Omega (\Phi_\alpha(u_0) + \Phi_\varphi(v_0))\,\mathrm{d}x + (1+a)\int_0^t\int_\Omega (u_\mu^\varepsilon(\alpha^{-1}(u_\mu^\varepsilon))^{p-1} + v_\mu^\varepsilon(\varphi^{-1}(v_\mu^\varepsilon))^{p-1})\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned} \tag{31}$$

where $z_\mu^\varepsilon = u_\mu^\varepsilon + v_\mu^\varepsilon$. We used the integration by parts formula, the non-negativity of $\alpha^{-1}(u_\mu^\varepsilon)$ and $\varphi^{-1}(v_\mu^\varepsilon)$, and the estimate (11) stating that $w_\mu^\varepsilon \leq 1$ to get (31). Since $p \geq 2$ and the functions α^{-1}

and φ^{-1} are nonnegative and nondecreasing, the second and third integrals in (31) are nonnegative. In view of (19) and the fact that $\Phi_\alpha(u^\varepsilon_\mu) \geq 0$ we deduce from (31) that

$$\int_\Omega \Phi_\varphi(v^\varepsilon_\mu(t))\,\mathrm{d}x \leq \int_\Omega (\Phi_\alpha(u_0) + \Phi_\varphi(v_0))\,\mathrm{d}x \\ + (1+a)\int_0^t \int_\Omega (u^\varepsilon_\mu(\alpha^{-1}(u^\varepsilon_\mu))^{p-1} + v^\varepsilon_\mu(\varphi^{-1}(v^\varepsilon_\mu))^{p-1})\,\mathrm{d}x\,\mathrm{d}s.$$

Next, it follows from (22) and (24) that

$$\int_\Omega \Phi_\varphi(v^\varepsilon_\mu(t))\,\mathrm{d}x \leq C_3 + (1+a)C_4 t + (1+a)p\int_0^t \int_\Omega \Phi_\varphi(v^\varepsilon_\mu)\,\mathrm{d}x\,\mathrm{d}s. \tag{32}$$

Gronwall's inequality (see Lemma 4.1.2 on p. 169 in [11]) implies for $t > 0$ that

$$\int_\Omega \Phi_\varphi(v^\varepsilon_\mu(t))\,\mathrm{d}x \leq \left(C_3 + \frac{C_4}{p}\right) e^{p(1+a)t}, \tag{33}$$

where C_3 and C_4 are defined by (23) and (26), respectively. In view of (21) we finally deduce that

$$\frac{1}{p}\int_\Omega (v^\varepsilon_\mu(t))^p\,\mathrm{d}x \leq \frac{|\Omega|}{p}\left(((1+a)+\tilde{C})^p + C_2^{p-1}(1+a)^p + C_u(\alpha^{-1}(C_u))^{p-1}\right) e^{p(1+a)t}.$$

Thus, for each $t \in [0,T]$

$$\left\|v^\varepsilon_\mu(\cdot,t)\right\|_{L^p(\Omega)} \leq |\Omega|^{1/p}\left(((1+a)+\tilde{C})^p + C_2^{p-1}(1+a)^p + C_u(\alpha^{-1}(C_u))^{p-1}\right)^{1/p} e^{(1+a)t} \\ \leq \max\{1,|\Omega|\}e^{(1+a)T}\left(\sum_{i=1}^{3}\frac{\alpha_i^p}{\beta_i}\right)^{\frac{1}{p}},$$

where $\alpha_1 = (1+a)+\tilde{C}$, $\beta_1 = 1$, $\alpha_2 = C_2(1+a)$, $\beta_2 = C_2$, $\alpha_3 = \alpha^{-1}(C_u)$ and $\beta_3 = \alpha^{-1}(C_u)/C_u$. The constants α_i and β_i for $i = 1,2,3$ are nonnegative. Thus, we deduce that for each $p \geq 2$

$$\left\|v^\varepsilon_\mu(\cdot,t)\right\|_{L^p(\Omega)} \leq \max\{1,|\Omega|\}e^{(1+a)T}\max\left\{1,\frac{3}{\beta_j}\right\}\alpha_j,$$

for some $j \in \{1,2,3\}$. Hence, the L^p-norm of $v^\varepsilon_\mu(\cdot,t)$ is bounded uniformly in p for each $p \in [2,\infty)$ and $t \in [0,T]$. Consequently, see Theorems 3.10.7 and 3.10.8. on p. 81 in [14][1], $v^\varepsilon_\mu(\cdot,t) \in L^\infty(\Omega)$ for $t \in [0,T]$. Because of the regularity of v^ε_μ we conclude the proof of (30). □

By repeating the proof of Corollary 4.5 for $p = 2$ we get important estimates of the gradients of u^ε_μ and v^ε_μ.

Corollary 4.6. *Suppose that (H_φ) and $(H_0)(i)$ are satisfied. Then, there exists a constant C_5 independent of ε and μ such that*

$$\mu\int_0^T \int_\Omega \left|\nabla u^\varepsilon_\mu\right|^2 \mathrm{d}x\,\mathrm{d}t \leq C_5, \tag{34}$$

$$d\int_0^T \int_\Omega \left|\nabla v^\varepsilon_\mu\right|^2 \mathrm{d}x\,\mathrm{d}t \leq C_5. \tag{35}$$

[1] Theorems 3.10.7 and 3.10.8. in [14]: Let $|\Omega| < \infty$. Let $1 \leq p_1 \leq p_2 \leq \ldots$ and suppose that $\lim_{k\to\infty} p_k = \infty$. Let $f \in \bigcap_{k=1}^\infty L^{p_k}(\Omega)$ and $a = \sup_{k\in\mathbb{N}}\|f\|_{L^{p_k}} < \infty$. Then $f \in L^\infty(\Omega)$ and $\|f\|_{L^\infty(\Omega)} = \lim_{p\to\infty}\|f\|_{L^p(\Omega)}$.

Proof. Since $\Phi_\alpha(u_\mu^\varepsilon) \geq 0$ and $\Phi_\varphi(v_\mu^\varepsilon) \geq 0$ for $u_\mu^\varepsilon \geq 0$ and $v_\mu^\varepsilon \geq 0$, respectively, it follows from (31) for $p = 2$ and the estimates (19) and (22) that

$$\mu \int_0^T \int_\Omega (\alpha^{-1})'(u_\mu^\varepsilon) \left|\nabla u_\mu^\varepsilon\right|^2 \mathrm{d}x\,\mathrm{d}t + d \int_0^T \int_\Omega (\varphi^{-1})'(v_\mu^\varepsilon) \left|\nabla v_\mu^\varepsilon\right|^2 \mathrm{d}x\,\mathrm{d}t$$
$$\leq C_3 + (1+a) \int_0^T \int_\Omega (u_\mu^\varepsilon \alpha^{-1}(u_\mu^\varepsilon) + v_\mu^\varepsilon \varphi^{-1}(v_\mu^\varepsilon))\,\mathrm{d}x\,\mathrm{d}t.$$

Since α^{-1} and φ^{-1} are nondecreasing functions, then in view of (18) and (30) we obtain

$$\mu \int_0^T \int_\Omega (\alpha^{-1})'(u_\mu^\varepsilon) \left|\nabla u_\mu^\varepsilon\right|^2 \mathrm{d}x\,\mathrm{d}t + d \int_0^T \int_\Omega (\varphi^{-1})'(v_\mu^\varepsilon) \left|\nabla v_\mu^\varepsilon\right|^2 \mathrm{d}x\,\mathrm{d}t \leq C_5, \tag{36}$$

where $C_5 = C_3 + (1+a)(C_u \alpha^{-1}(C_u) + C_v \varphi^{-1}(C_v))|\Omega| T$. Moreover, it follows from (H_φ) that

$$(\varphi^{-1})'(v_\mu^\varepsilon) = \frac{1}{\varphi'(\varphi^{-1}(v_\mu^\varepsilon))} > 1. \tag{37}$$

It follows also from (H_φ) that $\alpha'(s) = 1 - \varphi'(s) < 1$ for $s \geq 0$. Thus,

$$(\alpha^{-1})'(u_\mu^\varepsilon) = \frac{1}{\alpha'(\alpha^{-1}(u_\mu^\varepsilon))} > 1. \tag{38}$$

We deduce (34) and (35) from (36)–(38). □

Finally, we present a similar estimate for w_μ^ε.

Lemma 4.7. *There exists a constant C_6 independent of ε and μ such that*

$$\int_0^T \int_\Omega \left|\nabla w_\mu^\varepsilon\right|^2 \mathrm{d}x\,\mathrm{d}t \leq C_6. \tag{39}$$

Proof. Multiplying the equation for w_μ^ε by w_μ^ε and integrating in space yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (w_\mu^\varepsilon)^2 \mathrm{d}x + \int_\Omega \left|\nabla w_\mu^\varepsilon\right|^2 + b \int_\Omega (u_\mu^\varepsilon + v_\mu^\varepsilon)(w_\mu^\varepsilon)^2 = r \int_\Omega (1 - w_\mu^\varepsilon)(w_\mu^\varepsilon)^2 \leq r|\Omega|,$$

where we used the integration by parts formula and the uniform estimate (11). Integrating this inequality in time, (11) and $0 \leq w_0 \leq 1$ in $\overline{\Omega}$ implies the estimate (39) with $C_6 = (rT + 1/2)|\Omega|$. □

5 Relative compactness

In this section we always assume that (H_φ) and $(H_0)(i) - (ii)$ are satisfied. The system of equation $(\mathscr{P}_{\varepsilon,\mu})$ can be written in the form

$$(\mathscr{P}_{\varepsilon,\mu}) \begin{cases} u_t = \mu \Delta u + f_1(u,v,w) - \dfrac{1}{\varepsilon} F(u,v), \\ v_t = d \Delta v + f_2(u,v,w) + \dfrac{1}{\varepsilon} F(u,v), \\ w_t = \Delta w + f_3(u,v,w), \end{cases}$$

where we denote $f_1(u,v,w) = (1-u-v)u + auw$, $f_2(u,v,w) = (1-u-v)v + avw$, $f_3(u,v,w) = r(1-w)w - b(u+v)w$ and $F(u,v) = \varphi(u+v) - v$. We recall that $\mu < M$. In view of (H_φ) we have

$$F_u(u,v) = \varphi'(u+v) > 0 \qquad \text{and} \qquad F_v(u,v) = \varphi'(u+v) - 1 < 0,$$

in other words $F(u,v)$ is increasing in u and decreasing in v. From the monotonicity of F we deduce the following estimate.

Lemma 5.1. *Let $u, \widetilde{u}, v, \widetilde{v} \in \mathbb{R}$. Then*

$$(F(u,v) - F(\widetilde{u},\widetilde{v}))(\operatorname{sgn}(u-\widetilde{u}) - \operatorname{sgn}(v-\widetilde{v})) \geq 0. \tag{40}$$

Proof. Since $\operatorname{sgn}(u-\widetilde{u}) = \operatorname{sgn}(v-\widetilde{v})$ for $u > \widetilde{u}$ and $v > \widetilde{v}$, $u < \widetilde{u}$ and $v < \widetilde{v}$ and $u = \widetilde{u}$ and $v = \widetilde{v}$, the inequality (40) is trivially satisfied. Let $u > \widetilde{u}$ and $v \leq \widetilde{v}$. Then $\operatorname{sgn}(u-\widetilde{u}) - \operatorname{sgn}(v-\widetilde{v}) \geq 1$ and

$$F(u,v) - F(\widetilde{u},\widetilde{v}) = (F(u,v) - F(\widetilde{u},v)) + (F(\widetilde{u},v) - F(\widetilde{u},\widetilde{v})) \geq 0,$$

since both terms in the brackets are nonnegative due to the monotonicity of F. The remaining case $u \leq \widetilde{u}$ and $v > \widetilde{v}$ can be proved analogously. □

Next we prove two auxiliary lemmas.

Lemma 5.2. *There exists a constant C_7 such that*

$$\sum_{i=1}^{3} (f_i(u_1,u_2,u_3) - f_i(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3)) \operatorname{sgn}(u_i - \widetilde{u}_i) \leq C_7 \sum_{i=1}^{3} |u_i - \widetilde{u}_i| \tag{41}$$

for any two triples (u_1,u_2,u_3) and $(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3)$ such that $0 \leq u_1, \widetilde{u}_1 \leq C_u$, $0 \leq u_2, \widetilde{u}_2 \leq C_v$ and $0 \leq u_3, \widetilde{u}_3 \leq 1$.

Proof. The estimate (41) follows from the fact that the functions f_i, $i = 1, \ldots, 3$ are Lipschitz continuous on the domain $[0,C_u] \times [0,C_v] \times [0,1]$. □

Lemma 5.3. *There exists a constant C_8 such that for every $\tau \in (0,T)$*

$$\int_\Omega \left(\left| u_\mu^\varepsilon(\cdot,\tau) - u_0 \right| + \left| v_\mu^\varepsilon(\cdot,\tau) - v_0 \right| + \left| w_\mu^\varepsilon(\cdot,\tau) - w_0 \right| \right) \mathrm{d}x \leq C_8 \tau \tag{42}$$

where C_8 is independent of τ, ε and μ.

Proof. We multiply the equation for u_μ^ε by $\operatorname{sgn}_\eta(u_\mu^\varepsilon - u_0)$, where sgn_η is a smooth nondecreasing approximation of the sign function as already discussed above, and integrate in space over Ω to obtain that

$$\int_\Omega \frac{\partial (u_\mu^\varepsilon - u_0)}{\partial t} \operatorname{sgn}_\eta(u_\mu^\varepsilon - u_0)\, \mathrm{d}x - \mu \int_\Omega \Delta(u_\mu^\varepsilon - u_0) \operatorname{sgn}_\eta(u_\mu^\varepsilon - u_0)\, \mathrm{d}x + \frac{1}{\varepsilon} \int_\Omega F(u_\mu^\varepsilon, v_\mu^\varepsilon) \operatorname{sgn}_\eta(u_\mu^\varepsilon - u_0)\, \mathrm{d}x$$
$$= \mu \int_\Omega \Delta u_0 \operatorname{sgn}_\eta(u_\mu^\varepsilon - u_0)\, \mathrm{d}x + \int_\Omega f_1(u_\mu^\varepsilon, v_\mu^\varepsilon, w_\mu^\varepsilon) \operatorname{sgn}_\eta(u_\mu^\varepsilon - u_0)\, \mathrm{d}x$$

where we used $\partial_t u_\mu^\varepsilon = \partial_t(u_\mu^\varepsilon - u_0)$. Similarly we multiply the equation for v_μ^ε by $\operatorname{sgn}_\eta(v_\mu^\varepsilon - v_0)$ and integrate over Ω to obtain

$$\int_\Omega \frac{\partial(v_\mu^\varepsilon - v_0)}{\partial t}\operatorname{sgn}_\eta(v_\mu^\varepsilon - v_0)\,\mathrm{d}x - d\int_\Omega \Delta(v_\mu^\varepsilon - v_0)\operatorname{sgn}_\eta(v_\mu^\varepsilon - v_0)\,\mathrm{d}x - \frac{1}{\varepsilon}\int_\Omega F(u_\mu^\varepsilon, v_\mu^\varepsilon)\operatorname{sgn}_\eta(v_\mu^\varepsilon - v_0)\,\mathrm{d}x$$
$$= d\int_\Omega \Delta v_0 \operatorname{sgn}_\eta(v_\mu^\varepsilon - v_0)\,\mathrm{d}x + \int_\Omega f_2(u_\mu^\varepsilon, v_\mu^\varepsilon, w_\mu^\varepsilon)\operatorname{sgn}_\eta(v_\mu^\varepsilon - v_0)\,\mathrm{d}x.$$

Finally, we multiply the equation for w_μ^ε by $\operatorname{sgn}_\eta(w_\mu^\varepsilon - w_0)$ and integrate over Ω to obtain

$$\int_\Omega \frac{\partial(w_\mu^\varepsilon - w_0)}{\partial t}\operatorname{sgn}_\eta(w_\mu^\varepsilon - w_0)\,\mathrm{d}x - \int_\Omega \Delta(w_\mu^\varepsilon - w_0)\operatorname{sgn}_\eta(w_\mu^\varepsilon - w_0)\,\mathrm{d}x$$
$$= \int_\Omega \Delta w_0 \operatorname{sgn}_\eta(w_\mu^\varepsilon - w_0)\,\mathrm{d}x + \int_\Omega f_3(u_\mu^\varepsilon, v_\mu^\varepsilon, w_\mu^\varepsilon)\operatorname{sgn}_\eta(w_\mu^\varepsilon - w_0)\,\mathrm{d}x.$$

Integration by parts yields

$$-\mu\int_\Omega \Delta(u_\mu^\varepsilon - u_0)\operatorname{sgn}_\eta(u_\mu^\varepsilon - u_0)\,\mathrm{d}x = \mu\int_\Omega \left|\nabla(u_\mu^\varepsilon - u_0)\right|^2 \operatorname{sgn}'_\eta(u_\mu^\varepsilon - u_0)\,\mathrm{d}x, \tag{43}$$
$$-d\int_\Omega \Delta(v_\mu^\varepsilon - v_0)\operatorname{sgn}_\eta(v_\mu^\varepsilon - v_0)\,\mathrm{d}x = d\int_\Omega \left|\nabla(v_\mu^\varepsilon - v_0)\right|^2 \operatorname{sgn}'_\eta(v_\mu^\varepsilon - v_0)\,\mathrm{d}x \tag{44}$$

and

$$-\int_\Omega \Delta(w_\mu^\varepsilon - w_0)\operatorname{sgn}_\eta(w_\mu^\varepsilon - w_0)\,\mathrm{d}x = \int_\Omega \left|\nabla(w_\mu^\varepsilon - w_0)\right|^2 \operatorname{sgn}'_\eta(w_\mu^\varepsilon - w_0)\,\mathrm{d}x. \tag{45}$$

Since sgn_η is a nondecreasing function, the integrals (43)–(45) are nonnegative. In view of the assumption that $u_0, v_0, w_0 \in W^{2,1}(\Omega)$ and since $\mu < M$, there exists a constant $C_1 > 0$ independent of ε and μ such that

$$\mu\int_\Omega \Delta u_0 \operatorname{sgn}_\eta(u_\mu^\varepsilon - u_0)\,\mathrm{d}x \le \mu\|\Delta u_0\|_{L^1(\Omega)} \le C_1,$$
$$d\int_\Omega \Delta v_0 \operatorname{sgn}_\eta(v_\mu^\varepsilon - v_0)\,\mathrm{d}x \le d\|\Delta v_0\|_{L^1(\Omega)} \le C_1$$

and

$$\int_\Omega \Delta w_0 \operatorname{sgn}_\eta(w_\mu^\varepsilon - w_0)\,\mathrm{d}x \le \|\Delta w_0\|_{L^1(\Omega)} \le C_1.$$

Moreover, in view of the L^∞ estimates (11), (18) and (30) we deduce that there exists a constant $C_9 > 0$ independent of ε and μ such that

$$\int_\Omega f_1(u_\mu^\varepsilon, v_\mu^\varepsilon, w_\mu^\varepsilon)\operatorname{sgn}_\eta(u_\mu^\varepsilon - u_0)\,\mathrm{d}x \le C_9, \qquad \int_\Omega f_2(u_\mu^\varepsilon, v_\mu^\varepsilon, w_\mu^\varepsilon)\operatorname{sgn}_\eta(v_\mu^\varepsilon - v_0)\,\mathrm{d}x \le C_9$$

and

$$\int_\Omega f_3(u_\mu^\varepsilon, v_\mu^\varepsilon, w_\mu^\varepsilon)\operatorname{sgn}_\eta(w_\mu^\varepsilon - w_0)\,\mathrm{d}x \le C_9.$$

Thus, we deduce from the previous estimates that

$$\int_\Omega \frac{\partial(u_\mu^\varepsilon - u_0)}{\partial t}\operatorname{sgn}_\eta(u_\mu^\varepsilon - u_0)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega F(u_\mu^\varepsilon, v_\mu^\varepsilon)\operatorname{sgn}_\eta(u_\mu^\varepsilon - u_0)\,\mathrm{d}x \le C_1 + C_9,$$

$$\int_\Omega \frac{\partial(v_\mu^\varepsilon - v_0)}{\partial t}\,\mathrm{sgn}_\eta(v_\mu^\varepsilon - v_0)\,\mathrm{d}x - \frac{1}{\varepsilon}\int_\Omega F(u_\mu^\varepsilon, v_\mu^\varepsilon)\,\mathrm{sgn}_\eta(v_\mu^\varepsilon - v_0)\,\mathrm{d}x \le C_1 + C_9$$

and

$$\int_\Omega \frac{\partial(w_\mu^\varepsilon - w_0)}{\partial t}\,\mathrm{sgn}_\eta(w_\mu^\varepsilon - w_0)\,\mathrm{d}x \le C_1 + C_9.$$

Lebesgue's dominated convergence theorem yields in the limit $\eta \to 0$ that

$$\int_\Omega \frac{\partial(u_\mu^\varepsilon - u_0)}{\partial t}\,\mathrm{sgn}(u_\mu^\varepsilon - u_0)\,\mathrm{d}x + \int_\Omega \frac{\partial(v_\mu^\varepsilon - v_0)}{\partial t}\,\mathrm{sgn}(v_\mu^\varepsilon - v_0)\,\mathrm{d}x + \int_\Omega \frac{\partial(w_\mu^\varepsilon - w_0)}{\partial t}\,\mathrm{sgn}(w_\mu^\varepsilon - w_0)\,\mathrm{d}x$$
$$+\frac{1}{\varepsilon}\int_\Omega F(u_\mu^\varepsilon, v_\mu^\varepsilon)(\,\mathrm{sgn}(u_\mu^\varepsilon - u_0) - \mathrm{sgn}(v_\mu^\varepsilon - v_0))\,\mathrm{d}x \le 3(C_1 + C_9).$$

Thus,

$$\frac{d}{dt}\int_\Omega (|u_\mu^\varepsilon - u_0| + |v_\mu^\varepsilon - v_0| + |w_\mu^\varepsilon - w_0|)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega F(u_\mu^\varepsilon, v_\mu^\varepsilon)(\,\mathrm{sgn}(u_\mu^\varepsilon - u_0) - \mathrm{sgn}(v_\mu^\varepsilon - v_0))\,\mathrm{d}x$$
$$\le 3(C_1 + C_9).$$

Since $F(u_0, v_0) = 0$ by the assumption $(H_0)(ii)$, then by using Lemma 5.1 we obtain

$$F(u_\mu^\varepsilon, v_\mu^\varepsilon)(\,\mathrm{sgn}(u_\mu^\varepsilon - u_0) - \mathrm{sgn}(v_\mu^\varepsilon - v_0)) = (F(u_\mu^\varepsilon, v_\mu^\varepsilon) - F(u_0, v_0))(\,\mathrm{sgn}(u_\mu^\varepsilon - u_0) - \mathrm{sgn}(v_\mu^\varepsilon - v_0)) \ge 0,$$

which implies that

$$\frac{d}{dt}\int_\Omega (|u_\mu^\varepsilon - u_0| + |v_\mu^\varepsilon - v_0^\varepsilon| + |w_\mu^\varepsilon - w_0|)\,\mathrm{d}x \le 3(C_1 + C_9). \tag{46}$$

Integrating (46) in time t over $(0, \tau)$ implies (42) for $C_8 = 3(C_1 + C_9)$. □

Now we prove that the time translates of $u_\mu^\varepsilon, v_\mu^\varepsilon$ and w_μ^ε are bounded in the L^1 norm uniformly in the translation parameter τ.

Lemma 5.4. *There exists a constant C_{10} such that for every $\tau \in (0, T)$*

$$\int_0^{T-\tau}\int_\Omega |u_\mu^\varepsilon(x, t+\tau) - u_\mu^\varepsilon(x,t)|\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T-\tau}\int_\Omega |v_\mu^\varepsilon(x, t+\tau) - v_\mu^\varepsilon(x,t)|\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^{T-\tau}\int_\Omega |w_\mu^\varepsilon(x, t+\tau) - w_\mu^\varepsilon(x,t)|\,\mathrm{d}x\,\mathrm{d}t \le C_{10}\tau \tag{47}$$

where C_{10} is independent of τ, ε and μ.

Proof. Let $u_\tau(x,t) = u_\mu^\varepsilon(x, t+\tau)$, $v_\tau(x,t) = v_\mu^\varepsilon(x, t+\tau)$ and $w_\tau(x,t) = w_\mu^\varepsilon(x, t+\tau)$ and let $u(x,t) = u_\mu^\varepsilon(x,t)$, $v(x,t) = v_\mu^\varepsilon(x,t)$ and $w(x,t) = w_\mu^\varepsilon(x,t)$. Multiplying the equation for $u_\tau - u$ by $\mathrm{sgn}_\eta(u_\tau - u)$ and integrating over Ω yield

$$\int_\Omega \frac{\partial(u_\tau - u)}{\partial t}\,\mathrm{sgn}_\eta(u_\tau - u)\,\mathrm{d}x - \mu\int_\Omega \Delta(u_\tau - u)\,\mathrm{sgn}_\eta(u_\tau - u)\,\mathrm{d}x$$
$$+\frac{1}{\varepsilon}\int_\Omega (F(u_\tau, v_\tau) - F(u,v))\,\mathrm{sgn}_\eta(u_\tau - u)\,\mathrm{d}x = \int_\Omega (f_1(u_\tau, v_\tau, w_\tau) - f_1(u,v,w))\,\mathrm{sgn}_\eta(u_\tau - u)\,\mathrm{d}x.$$

Similarly, by multiplying the equation for $v_\tau - v$ by $\operatorname{sgn}_\eta(v_\tau - v)$ and integrating over Ω we obtain

$$\int_\Omega \frac{\partial(v_\tau - v)}{\partial t} \operatorname{sgn}_\eta(v_\tau - v)\,\mathrm{d}x - d\int_\Omega \Delta(v_\tau - v)\operatorname{sgn}_\eta(v_\tau - v)\,\mathrm{d}x$$
$$-\frac{1}{\varepsilon}\int_\Omega (F(u_\tau, v_\tau) - F(u,v))\operatorname{sgn}_\eta(v_\tau - v)\,\mathrm{d}x = \int_\Omega (f_2(u_\tau, v_\tau, w_\tau) - f_2(u,v,w))\operatorname{sgn}_\eta(v_\tau - v)\,\mathrm{d}x.$$

Finally, by multiplying the equation for $w_\tau - w$ by $\operatorname{sgn}_\eta(w_\tau - w)$ and integrating over Ω we obtain

$$\int_\Omega \frac{\partial(w_\tau - w)}{\partial t} \operatorname{sgn}_\eta(w_\tau - w)\,\mathrm{d}x - \int_\Omega \Delta(w_\tau - w)\operatorname{sgn}_\eta(w_\tau - w)\,\mathrm{d}x$$
$$= \int_\Omega (f_3(u_\tau, v_\tau, w_\tau) - f_3(u,v,w))\operatorname{sgn}_\eta(w_\tau - w)\,\mathrm{d}x.$$

As in the proof of Lemma 5.3, integration by parts yields that

$$-\mu\int_\Omega \Delta(u_\tau - u)\operatorname{sgn}_\eta(u_\tau - u)\,\mathrm{d}x = \mu\int_\Omega |\nabla(u_\tau - u)|^2\operatorname{sgn}'_\eta(u_\tau - u)\,\mathrm{d}x \geq 0,$$
$$-d\int_\Omega \Delta(v_\tau - v)\operatorname{sgn}_\eta(v_\tau - v)\,\mathrm{d}x = d\int_\Omega |\nabla(v_\tau - v)|^2\operatorname{sgn}'_\eta(v_\tau - v)\,\mathrm{d}x \geq 0$$

and

$$-\int_\Omega \Delta(w_\tau - w)\operatorname{sgn}_\eta(w_\tau - w)\,\mathrm{d}x = \int_\Omega |\nabla(w_\tau - w)|^2\operatorname{sgn}'_\eta(w_\tau - w)\,\mathrm{d}x \geq 0.$$

By using Lebesgue's dominated convergence theorem we obtain that, as $\eta \to 0$,

$$\int_\Omega \frac{\partial(u_\tau - u)}{\partial t}\operatorname{sgn}(u_\tau - u)\,\mathrm{d}x + \int_\Omega \frac{\partial(v_\tau - v)}{\partial t}\operatorname{sgn}(v_\tau - v)\,\mathrm{d}x + \int_\Omega \frac{\partial(w_\tau - w)}{\partial t}\operatorname{sgn}(w_\tau - w)\,\mathrm{d}x$$
$$+\frac{1}{\varepsilon}\int_\Omega (F(u_\tau, v_\tau) - F(u,v))(\operatorname{sgn}(u_\tau - u) - \operatorname{sgn}(v_\tau - v))\,\mathrm{d}x$$
$$\leq \int_\Omega \{(f_1(u_\tau, v_\tau, w_\tau) - f_1(u,v,w))\operatorname{sgn}(u_\tau - u) + (f_2(u_\tau, v_\tau, w_\tau) - f_2(u,v,w))\operatorname{sgn}(v_\tau - v)$$
$$+(f_3(u_\tau, v_\tau, w_\tau) - f_3(u,v,w))\operatorname{sgn}(w_\tau - w)\}\,\mathrm{d}x,$$

where we used $(F(u_\tau, v_\tau) - F(u,v))(\operatorname{sgn}(u_\tau - u) - \operatorname{sgn}(v_\tau - v)) \geq 0$. In view of Lemmas 5.1 and 5.2 we deduce that

$$\frac{d}{dt}\int_\Omega (|u_\tau - u| + |v_\tau - v| + |w_\tau - w|)\,\mathrm{d}x \leq C_7\int_\Omega (|u_\tau - u| + |v_\tau - v| + |w_\tau - w|)\,\mathrm{d}x.$$

Gronwall's inequality and (42) imply

$$\int_\Omega (|u_\tau(t) - u(t)| + |v_\tau(t) - v(t)| + |w_\tau(t) - w(t)|)\,\mathrm{d}x$$
$$\leq \left(\int_\Omega (|u_\tau(0) - u(0)| + |v_\tau(0) - v(0)| + |w_\tau(0) - w(0)|)\,\mathrm{d}x\right) e^{C_7 t}$$
$$\leq C_8 e^{C_7 T}\tau. \tag{48}$$

Finally, we integrate the last inequality over $(0, T-\tau)$ to obtain that

$$\int_0^{T-\tau} \int_\Omega (|u_\tau(t)-u(t)| + |v_\tau(t)-v(t)| + |w_\tau(t)-w(t)|)\,\mathrm{d}x\,\mathrm{d}t \le C_{10}\tau,$$

where $C_{10} = C_8 T e^{C_7 T}$. □

We prove that the space translates of $u_\mu^\varepsilon, v_\mu^\varepsilon$ and w_μ^ε are bounded in the L^1 norm uniformly in the translation parameter ξ. We need the following lemma.

Lemma 5.5. *For each $r \in (0, \hat{r})$ and $\hat{r} > 0$ sufficiently small it holds that*

$$\mu \int_0^T \int_{\Omega_r'} (u_\mu^\varepsilon(x+\xi,t) - u_\mu^\varepsilon(x,t))^2\,\mathrm{d}x\,\mathrm{d}t \le C_5|\xi|^2, \tag{49}$$

$$d \int_0^T \int_{\Omega_r'} (v_\mu^\varepsilon(x+\xi,t) - v_\mu^\varepsilon(x,t))^2\,\mathrm{d}x\,\mathrm{d}t \le C_5|\xi|^2, \tag{50}$$

$$\int_0^T \int_{\Omega_r'} (w_\mu^\varepsilon(x+\xi,t) - w_\mu^\varepsilon(x,t))^2\,\mathrm{d}x\,\mathrm{d}t \le C_6|\xi|^2 \tag{51}$$

for all $\xi \in \mathbb{R}^N$ such that $|\xi| \le r$ where the constants C_5 and C_6 are from Corollary 4.6 and Lemma 4.7, respectively.

Proof. By direct calculations we get

$$\begin{aligned}
\mu \int_0^T \int_{\Omega_r'} (u_\mu^\varepsilon(x+\xi,t) - u_\mu^\varepsilon(x,t))^2\,\mathrm{d}x\,\mathrm{d}t &= \mu \int_0^T \int_{\Omega_r'} \left(\int_0^1 \frac{\partial}{\partial\theta} u_\mu^\varepsilon(x+\theta\xi,t)\mathrm{d}\theta \right)^2 \mathrm{d}x\,\mathrm{d}t \\
&= \mu \int_0^T \int_{\Omega_r'} \left(\int_0^1 \nabla u_\mu^\varepsilon(x+\theta\xi,t)\cdot\xi\,\mathrm{d}\theta \right)^2 \mathrm{d}x\,\mathrm{d}t \\
&\le \mu|\xi|^2 \int_0^1 \int_0^T \int_{\Omega_r'} \left|\nabla u_\mu^\varepsilon(x+\theta\xi,t)\right|^2 \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\theta \\
&\le \mu|\xi|^2 \int_0^T \int_\Omega \left|\nabla u_\mu^\varepsilon(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\le C_5|\xi|^2
\end{aligned}$$

due to the Hölder inequality and (34). Analogously we prove (50) and (51). □

Lemma 5.6. *For each $r \in (0, \hat{r})$ and $\hat{r} > 0$ sufficiently small, there exists a positive function $\rho(\xi)$ such that $\rho(\xi) \to 0$ uniformly in ε and μ as $|\xi| \to 0$ and*

$$\begin{aligned}
\int_0^T \int_{\Omega_r} \left|u_\mu^\varepsilon(x+\xi,t) - u_\mu^\varepsilon(x,t)\right| \mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega_r} \left|v_\mu^\varepsilon(x+\xi,t) - v_\mu^\varepsilon(x,t)\right| \mathrm{d}x\,\mathrm{d}t \\
+ \int_0^T \int_{\Omega_r} \left|w_\mu^\varepsilon(x+\xi,t) - w_\mu^\varepsilon(x,t)\right| \mathrm{d}x\,\mathrm{d}t \le \rho(\xi)
\end{aligned} \tag{52}$$

for all $\xi \in \mathbb{R}^N$ such that $|\xi| \le r$.

Proof. Let $u_\xi = u_\mu^\varepsilon(x+\xi,t), v_\xi = v_\mu^\varepsilon(x+\xi,t)$ and $w_\xi = w_\mu^\varepsilon(x+\xi,t)$ and let $u = u_\mu^\varepsilon(x,t), v = v_\mu^\varepsilon(x,t)$ and $w = w_\mu^\varepsilon(x,t)$. We start by multiplying the equation for $u_\xi - u$ by $\operatorname{sgn}_\eta(u_\xi - u)\psi$, where ψ is defined by (16), and integrating over $\Omega_r' \times (0,t)$ to obtain

$$\int_0^t \int_{\Omega_r'} \frac{\partial(u_\xi - u)}{\partial t} \operatorname{sgn}_\eta(u_\xi - u)\psi \,\mathrm{d}x\mathrm{d}s + \frac{1}{\varepsilon}\int_0^t \int_{\Omega_r'} (F(u_\xi,v_\xi) - F(u,v)) \operatorname{sgn}_\eta(u_\xi - u)\psi \,\mathrm{d}x\mathrm{d}s$$
$$= \mu \int_0^t \int_{\Omega_r'} \Delta(u_\xi - u) \operatorname{sgn}_\eta(u_\xi - u)\psi \,\mathrm{d}x\mathrm{d}s + \int_0^t \int_{\Omega_r'} (f_1(u_\xi,v_\xi,w_\xi) - f_1(u,v,w)) \operatorname{sgn}_\eta(u_\xi - u)\psi \,\mathrm{d}x\mathrm{d}s.$$

Similarly, we deduce from the equations for $v_\xi - v$ and $w_\xi - w$ that

$$\int_0^t \int_{\Omega_r'} \frac{\partial(v_\xi - v)}{\partial t} \operatorname{sgn}_\eta(v_\xi - v)\psi \,\mathrm{d}x\mathrm{d}s - \frac{1}{\varepsilon}\int_0^t \int_{\Omega_r'} (F(u_\xi,v_\xi) - F(u,v)) \operatorname{sgn}_\eta(v_\xi - v)\psi \,\mathrm{d}x\mathrm{d}s$$
$$= d \int_0^t \int_{\Omega_r'} \Delta(v_\xi - v) \operatorname{sgn}_\eta(v_\xi - v)\psi \,\mathrm{d}x\mathrm{d}s + \int_0^t \int_{\Omega_r'} (f_2(u_\xi,v_\xi,w_\xi) - f_2(u,v,w)) \operatorname{sgn}_\eta(v_\xi - v)\psi \,\mathrm{d}x\mathrm{d}s$$

and

$$\int_0^t \int_{\Omega_r'} \frac{\partial(w_\xi - w)}{\partial t} \operatorname{sgn}_\eta(w_\xi - w)\psi \,\mathrm{d}x\mathrm{d}s = \int_0^t \int_{\Omega_r'} \Delta(w_\xi - w) \operatorname{sgn}_\eta(w_\xi - w)\psi \,\mathrm{d}x\mathrm{d}s$$
$$+ \int_0^t \int_{\Omega_r'} (f_3(u_\xi,v_\xi,w_\xi) - f_3(u,v,w)) \operatorname{sgn}_\eta(w_\xi - w)\psi \,\mathrm{d}x\mathrm{d}s.$$

The integrals with Laplacians can be further handled by the integration by parts formula. Thus, for example,

$$\begin{aligned}
&\mu \int_0^t \int_{\Omega_r'} \Delta(u_\xi - u) \operatorname{sgn}_\eta(u_\xi - u)\psi \,\mathrm{d}x\mathrm{d}s \\
&= -\mu \int_0^t \int_{\Omega_r'} \nabla(u_\xi - u) \cdot \nabla(\operatorname{sgn}_\eta(u_\xi - u)\psi) \,\mathrm{d}x\mathrm{d}s \\
&= -\mu \int_0^t \int_{\Omega_r'} \left|\nabla(u_\xi - u)\right|^2 \operatorname{sgn}_\eta'(u_\xi - u)\psi \,\mathrm{d}x\mathrm{d}s - \mu \int_0^t \int_{\Omega_r'} \operatorname{sgn}_\eta(u_\xi - u)\nabla(u_\xi - u) \cdot \nabla\psi \,\mathrm{d}x\mathrm{d}s \\
&\le -\mu \int_0^t \int_{\Omega_r'} \operatorname{sgn}_\eta(u_\xi - u)\nabla(u_\xi - u) \cdot \nabla\psi \,\mathrm{d}x\mathrm{d}s
\end{aligned}$$

where the boundary integral is equal to zero due to vanishing ψ at the boundary of Ω_r' and the inequality follows from the fact that the first integral on the third line is nonnegative. Similarly, we obtain

$$d \int_0^t \int_{\Omega_r'} \Delta(v_\xi - v) \operatorname{sgn}_\eta(v_\xi - v)\psi \,\mathrm{d}x\mathrm{d}s \le -d \int_0^t \int_{\Omega_r'} \operatorname{sgn}_\eta(v_\xi - v)\nabla(v_\xi - v) \cdot \nabla\psi \,\mathrm{d}x\mathrm{d}s,$$
$$\int_0^t \int_{\Omega_r'} \Delta(w_\xi - w) \operatorname{sgn}_\eta(w_\xi - w)\psi \,\mathrm{d}x\mathrm{d}s \le -\int_0^t \int_{\Omega_r'} \operatorname{sgn}_\eta(w_\xi - w)\nabla(w_\xi - w) \cdot \nabla\psi \,\mathrm{d}x\mathrm{d}s.$$

Next we apply Lebesgue's dominated convergence theorem to deduce from the equation for $u_\xi - u$ that

$$\int_0^t \int_{\Omega_r'} \frac{\partial(u_\xi - u)}{\partial t} \operatorname{sgn}(u_\xi - u)\psi \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{\varepsilon}\int_0^t \int_{\Omega_r'} (F(u_\xi, v_\xi) - F(u,v)) \operatorname{sgn}(u_\xi - u)\psi \,\mathrm{d}x\,\mathrm{d}s$$
$$\leq -\mu \int_0^t \int_{\Omega_r'} \operatorname{sgn}(u_\xi - u)\nabla(u_\xi - u)\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t \int_{\Omega_r'} (f_1(u_\xi, v_\xi, w_\xi) - f_1(u,v,w)) \operatorname{sgn}(u_\xi - u)\psi \,\mathrm{d}x\,\mathrm{d}s,$$

from the equation for $v_\xi - v$ that

$$\int_0^t \int_{\Omega_r'} \frac{\partial(v_\xi - v)}{\partial t} \operatorname{sgn}(v_\xi - v)\psi \,\mathrm{d}x\,\mathrm{d}s - \frac{1}{\varepsilon}\int_0^t \int_{\Omega_r'} (F(u_\xi, v_\xi) - F(u,v)) \operatorname{sgn}(v_\xi - v)\psi \,\mathrm{d}x\,\mathrm{d}s$$
$$\leq -d \int_0^t \int_{\Omega_r'} \operatorname{sgn}(v_\xi - v)\nabla(v_\xi - v)\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t \int_{\Omega_r'} (f_2(u_\xi, v_\xi, w_\xi) - f_2(u,v,w)) \operatorname{sgn}(v_\xi - v)\psi \,\mathrm{d}x\,\mathrm{d}s$$

and from the equation for $w_\xi - w$ that

$$\int_0^t \int_{\Omega_r'} \frac{\partial(w_\xi - w)}{\partial t} \operatorname{sgn}(w_\xi - w)\psi \,\mathrm{d}x\,\mathrm{d}s \leq -\int_0^t \int_{\Omega_r'} \operatorname{sgn}(w_\xi - w)\nabla(w_\xi - w)\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t \int_{\Omega_r'} (f_3(u_\xi, v_\xi, w_\xi) - f_3(u,v,w)) \operatorname{sgn}(w_\xi - w)\psi \,\mathrm{d}x\,\mathrm{d}s.$$

The integration by parts formula, the Hölder inequality and (49) imply that

$$-\mu \int_0^t \int_{\Omega_r'} \operatorname{sgn}(u_\xi - u)\nabla(u_\xi - u)\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}s = -\mu \int_0^t \int_{\Omega_r'} \nabla\big|u_\xi - u\big|\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}s$$
$$= \mu \int_0^t \int_{\Omega_r'} \big|u_\xi - u\big|\Delta\psi \,\mathrm{d}x\,\mathrm{d}s \leq \mu\sqrt{T}\big\|u_\xi - u\big\|_{L^2(\Omega_r'\times(0,T))}\|\Delta\psi\|_{L^2(\Omega_r')} \leq \sqrt{\mu T C_5}\|\Delta\psi\|_{L^2(\Omega_r')}|\xi|.$$

Moreover, since $\mu < M$, then

$$-\mu \int_0^t \int_{\Omega_r'} \operatorname{sgn}(u_\xi - u)\nabla(u_\xi - u)\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}s < \sqrt{M T C_5}\|\Delta\psi\|_{L^2(\Omega_r')}|\xi|.$$

In view of (50) and (51) we deduce in a similar way that

$$-d \int_0^t \int_{\Omega_r'} \operatorname{sgn}(v_\xi - v)\nabla(v_\xi - v)\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}s \leq \sqrt{d T C_5}\|\Delta\psi\|_{L^2(\Omega_r')}|\xi|$$

and

$$-\int_0^t \int_{\Omega_r'} \operatorname{sgn}(w_\xi - w)\nabla(w_\xi - w)\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}s \leq \sqrt{T C_6}\|\Delta\psi\|_{L^2(\Omega_r')}|\xi|.$$

Next we set $C_{11} = \sqrt{MTC_5}\|\Delta\psi\|_{L^2(\Omega_r')}$, $C_{12} = \sqrt{dTC_5}\|\Delta\psi\|_{L^2(\Omega_r')}$ and $C_{13} = \sqrt{TC_6}\|\Delta\psi\|_{L^2(\Omega_r')}$. Thus,

$$\begin{aligned}\int_0^t \int_{\Omega_r'} \frac{\partial(u_\xi - u)}{\partial t}\operatorname{sgn}(u_\xi - u)\psi\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{\varepsilon}\int_0^t \int_{\Omega_r'} (F(u_\xi,v_\xi) - F(u,v))\operatorname{sgn}(u_\xi - u)\psi\,\mathrm{d}x\,\mathrm{d}s \\ \le C_{11}|\xi| + \int_0^t \int_{\Omega_r'} (f_1(u_\xi,v_\xi,w_\xi) - f_1(u,v,w))\operatorname{sgn}(u_\xi - u)\psi\,\mathrm{d}x\,\mathrm{d}s,\end{aligned} \tag{53}$$

as well as

$$\begin{aligned}\int_0^t \int_{\Omega_r'} \frac{\partial(v_\xi - v)}{\partial t}\operatorname{sgn}(v_\xi - v)\psi\,\mathrm{d}x\,\mathrm{d}s - \frac{1}{\varepsilon}\int_0^t \int_{\Omega_r'} (F(u_\xi,v_\xi) - F(u,v))\operatorname{sgn}(v_\xi - v)\psi\,\mathrm{d}x\,\mathrm{d}s \\ \le C_{12}|\xi| + \int_0^t \int_{\Omega_r'} (f_2(u_\xi,v_\xi,w_\xi) - f_2(u,v,w))\operatorname{sgn}(v_\xi - v)\psi\,\mathrm{d}x\,\mathrm{d}s\end{aligned} \tag{54}$$

and

$$\begin{aligned}\int_0^t \int_{\Omega_r'} \frac{\partial(w_\xi - w)}{\partial t}\operatorname{sgn}(w_\xi - w)\psi\,\mathrm{d}x\,\mathrm{d}s \le C_{13}|\xi| \\ + \int_0^t \int_{\Omega_r'} (f_3(u_\xi,v_\xi,w_\xi) - f_3(u,v,w))\operatorname{sgn}(w_\xi - w)\psi\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{55}$$

In view of Lemmas 5.1 and 5.2, summing up (53)–(55) yields

$$\begin{aligned}\int_0^t \int_{\Omega_r'} \frac{\partial}{\partial t}\left(|u_\xi - u| + |v_\xi - v| + |w_\xi - w|\right)\psi\,\mathrm{d}x\,\mathrm{d}s \le (C_{11} + C_{12} + C_{13})|\xi| \\ + C_7\int_0^t \int_{\Omega_r'} \left(|u_\xi - u| + |v_\xi - v| + |w_\xi - w|\right)\psi\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{56}$$

Moreover, the uniform continuity of the initial data u_0, v_0 and w_0 in Ω_r' implies that there exists a positive function ω such that $\omega(\xi) \to 0$ as $|\xi| \to 0$ and

$$\int_{\Omega_r'} (|u_0(x+\xi) - u_0(x)| + |v_0(x+\xi) - v_0(x)| + |w_0(x+\xi) - w_0(x)|)\,\psi\,\mathrm{d}x \le \omega(\xi).$$

Thus, we obtain from (56) that

$$\begin{aligned}\int_{\Omega_r'} \left(|u_\xi(t) - u(t)| + |v_\xi(t) - v(t)| + |w_\xi(t) - w(t)|\right)\psi\,\mathrm{d}x \le \omega(\xi) + (C_{11} + C_{12} + C_{13})|\xi| \\ + C_7\int_0^t \int_{\Omega_r'} \left(|u_\xi(s) - u(s)| + |v_\xi(s) - v(s)| + |w_\xi(s) - w(s)|\right)\psi\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Finally, we deduce from Gronwall's inequality that

$$\int_{\Omega_r'} \left(|u_\xi(t) - u(t)| + |v_\xi(t) - v(t)| + |w_\xi(t) - w(t)|\right)\psi\,\mathrm{d}x \le (\omega(\xi) + (C_{11} + C_{12} + C_{13})|\xi|)e^{C_7 T}. \tag{57}$$

Since $0 \le \psi \le 1$ in Ω_r', $\psi = 1$ in Ω_r and $\Omega_r \subset \Omega_r'$, we obtain (52) by integrating (57) in time over $(0,T)$ and setting $\rho(\xi) = (\omega(\xi) + (C_{11} + C_{12} + C_{13})|\xi|)Te^{C_7 T}$. □

In view of the uniform L^∞ estimates (11), (18) and (30) we can use (47) and (52) to show that the time and space translates converge to zero uniformly in ε and μ as the translation parameters tend to zero in the L^p norm for any $p \in [2,\infty)$.

Corollary 5.7. *Let $p \in [2,\infty)$. a) There exists a constant C_{14} such that for every $\tau \in (0,T)$*

$$\int_0^{T-\tau}\int_\Omega \left|u_\mu^\varepsilon(x,t+\tau)-u_\mu^\varepsilon(x,t)\right|^p \mathrm{d}x\,\mathrm{d}t + \int_0^{T-\tau}\int_\Omega \left|v_\mu^\varepsilon(x,t+\tau)-v_\mu^\varepsilon(x,t)\right|^p \mathrm{d}x\,\mathrm{d}t + \int_0^{T-\tau}\int_\Omega \left|w_\mu^\varepsilon(x,t+\tau)-w_\mu^\varepsilon(x,t)\right|^p \mathrm{d}x\,\mathrm{d}t \le C_{14}\tau, \tag{58}$$

where C_{14} depends on p but not on τ, ε and μ.

b) For each $r \in (0,\hat{r})$ and $\hat{r} > 0$ sufficiently small, there exists a positive function $\widetilde{\rho}(\xi)$ such that $\widetilde{\rho}(\xi) \to 0$ uniformly in ε and μ as $|\xi| \to 0$ and

$$\int_0^{T}\int_{\Omega_r} \left|u_\mu^\varepsilon(x+\xi,t)-u_\mu^\varepsilon(x,t)\right|^p \mathrm{d}x\,\mathrm{d}t + \int_0^{T}\int_{\Omega_r} \left|v_\mu^\varepsilon(x+\xi,t)-v_\mu^\varepsilon(x,t)\right|^p \mathrm{d}x\,\mathrm{d}t + \int_0^{T}\int_{\Omega_r} \left|w_\mu^\varepsilon(x+\xi,t)-w_\mu^\varepsilon(x,t)\right|^p \mathrm{d}x\,\mathrm{d}t \le \widetilde{\rho}(\xi) \tag{59}$$

for all $\xi \in \mathbb{R}^N$ such that $|\xi| \le r$.

Proof. It follows from the uniform L^∞ estimate (18) that

$$\begin{aligned}\int_0^{T-\tau}\int_\Omega &\left|u_\mu^\varepsilon(x,t+\tau)-u_\mu^\varepsilon(x,t)\right|^p \mathrm{d}x\,\mathrm{d}t \\ &= \int_0^{T-\tau}\int_\Omega \left|u_\mu^\varepsilon(x,t+\tau)-u_\mu^\varepsilon(x,t)\right|^{p-1}\left|u_\mu^\varepsilon(x,t+\tau)-u_\mu^\varepsilon(x,t)\right| \mathrm{d}x\,\mathrm{d}t \\ &\le (2C_u)^{p-1}\int_0^{T-\tau}\int_\Omega \left|u_\mu^\varepsilon(x,t+\tau)-u_\mu^\varepsilon(x,t)\right| \mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

In view of (47) we obtain the first part of (58). Analogously we estimate the remaining integrals. □

Corollary 5.8. *The sequences $\{u_\mu^\varepsilon\}$, $\{v_\mu^\varepsilon\}$ and $\{w_\mu^\varepsilon\}$ are relatively compact in $L^p(Q_T)$ for $p \in [1,\infty)$.*

Proof. The estimates (58) and (59) imply that the differences of space and time translates of $u_\mu^\varepsilon, v_\mu^\varepsilon$ and w_μ^ε tend to zero uniformly in ε and μ in the L^p topology as the translation parameters tend to zero.

Moreover, in view of (18) we have that

$$\int_{T-\tau}^{T}\int_\Omega (u_\mu^\varepsilon(x,t))^p \,\mathrm{d}x\,\mathrm{d}t \le C_u^p|\Omega|\tau \quad \text{and} \quad \int_0^T\int_{\Omega\setminus\Omega_r} (u_\mu^\varepsilon(x,t))^p \,\mathrm{d}x\,\mathrm{d}t \le C_u^p T S_2(r), \tag{60}$$

where $S_2(r) = \int_{\Omega\setminus\Omega_r} 1\,\mathrm{d}x$, and similar estimates hold for v_μ^ε and w_μ^ε. The Fréchet-Kolmogorov theorem 3.1 applied to $\{u_\mu^\varepsilon\}$, $\{v_\mu^\varepsilon\}$ and $\{w_\mu^\varepsilon\}$ yields that these sequences are relatively compact in $L^p(Q_T)$. □

6 Proof of main theorems

In this section we complete the proof of Theorems 2.3 and 2.4.

As in [7], it follows from Corollary 5.8 and the uniform L^∞-boundedness of the sequences $\{u^\varepsilon_\mu\}$, $\{v^\varepsilon_\mu\}$ and $\{w^\varepsilon_\mu\}$ that there exist subsequences $\{u^\varepsilon_{\mu_k}\}_{\mu_k>0}$, $\{v^\varepsilon_{\mu_k}\}_{\mu_k>0}$ and $\{w^\varepsilon_{\mu_k}\}_{\mu_k>0}$ and functions $u^\varepsilon, v^\varepsilon, w^\varepsilon \in L^\infty(Q_T)$ such that for $p \in [1,\infty)$

$$u^\varepsilon_{\mu_k} \to u^\varepsilon, \quad v^\varepsilon_{\mu_k} \to v^\varepsilon, \quad w^\varepsilon_{\mu_k} \to w^\varepsilon \quad \text{strongly in } L^p(Q_T) \text{ and a.e. in } Q_T$$

as $\mu_k \to 0$. The sequences $\{u^\varepsilon\}$, $\{v^\varepsilon\}$ and $\{w^\varepsilon\}$ are relatively compact in $L^p(Q_T)$ for each $p \in [1,\infty)$, as well as uniformly bounded in $L^\infty(Q_T)$. Indeed, the relative compactness of the sequences $\{u^\varepsilon\}$, $\{v^\varepsilon\}$ and $\{w^\varepsilon\}$ follows from Corollary 5.8 and a remark that all the estimates on space and time translates in Section 4 are uniform in ε. Moreover, the functions $u^\varepsilon, v^\varepsilon$ and w^ε are a unique weak solution of Problem $(\mathscr{P}_\varepsilon)$ in the sense that they satisfy (12)–(15).

Proof of Theorem 2.3: Corollary 5.8 and the uniform boundedness of the sequences $\{u^\varepsilon\}_{\varepsilon>0}$, $\{v^\varepsilon\}_{\varepsilon>0}$ and $\{w^\varepsilon\}_{\varepsilon>0}$ imply that there exist subsequences $\{u^{\varepsilon_k}\}_{\varepsilon_k>0}$, $\{v^{\varepsilon_k}\}_{\varepsilon_k>0}$ and $\{w^{\varepsilon_k}\}_{\varepsilon_k>0}$ and functions $u, v, w \in L^\infty(Q_T)$ such that (6) is satisfied. This completes the proof. □

We next go to the proof of Theorem 2.4. Let us denote

$$z^{\varepsilon_k} = u^{\varepsilon_k} + v^{\varepsilon_k} \quad \text{and} \quad z = u + v.$$

We deduce from Theorem 2.3 that for $p \in [1,\infty)$

$$z^{\varepsilon_k} \to z \quad \text{strongly in } L^p(Q_T) \text{ and a.e. in } Q_T$$

as $\varepsilon_k \to 0$.

Proof of Theorem 2.4: Since φ is Lipschitz continuous then the convergence (7) implies that $\varphi(z^{\varepsilon_k}) \to \varphi(z)$ strongly in $L^p(Q_T)$ for $p \in [1,\infty)$ and a.e. in Q_T as $\varepsilon_k \to 0$. By taking $\xi = \varepsilon_k \zeta$ in (14) for $\zeta \in C^{2,1}(\overline{Q}_T)$ and passing to the limit $\varepsilon_k \to 0$ we deduce that

$$\iint_{Q_T} (\varphi(z) - v)\, \zeta \,\mathrm{d}x\,\mathrm{d}t = 0$$

for all $\zeta \in C^{2,1}(\overline{Q}_T)$. Thus, $v = \varphi(z)$ a.e. in Q_T. We deduce from (13) and (14) that z^{ε_k} satisfies

$$-\int_\Omega z_0 \xi(0)\,\mathrm{d}x = \iint_{Q_T} (d v^{\varepsilon_k} \Delta\xi + ((1 - z^{\varepsilon_k}) z^{\varepsilon_k} + a z^{\varepsilon_k} w^{\varepsilon_k})\xi + z^{\varepsilon_k} \xi_t)\,\mathrm{d}x\,\mathrm{d}t, \tag{61}$$

where $z_0 = u_0 + v_0$ and $\xi \in C^{2,1}(\overline{Q}_T)$ is such that $\xi(x,T) = 0$ in Ω and $\partial_\nu \xi = 0$ on $\partial\Omega \times [0,T]$. Thus, in view of (6), (7) and the identity $v = \varphi(z)$ a.e. in Q_T we deduce (8) and (9) from (61) and (15) respectively, as $\varepsilon_k \to 0$.

It remains to show that $z, w \in C([0,T]; L^1(\Omega))$. In the view of (48), taking $\mu_k \to 0$ and $\varepsilon_k \to 0$ the continuity immediately follows from the Arzela-Ascoli Theorem. Since the uniqueness of the weak solution of Problem $(\mathscr{P})$ was proved in [6], the proof is complete. □

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