Large time behaviour of the solution of a nonlinear diffusion problem in anthropology

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This article is dedicated to Professor Claude-Michel Brauner at the occasion of his 70th birthday.

Abstract

In this article we consider a reaction-diffusion model for the spreading of farmers in Europe, which was occupied by hunter-gatherers; this process is known as the Neolithic agricultural revolution. The spreading of farmers is modelled by a nonlinear porous medium type diffusion equation which coincides with the singular limit of another model for the dispersal of farmers as a small parameter tends to zero. From the ecological viewpoint, the nonlinear diffusion takes into account the population density pressure of the farmers on their dispersal. The interaction between farmers and hunter-gatherers is of the Lotka-Volterra prey-predator type. We show the existence and uniqueness of a global in time solution and study its asymptotic behaviour as time tends to infinity.

Keywords: farmer-hunters model; reaction-diffusion system; degenerate diffusion; existence and uniqueness of the solution; exponential convergence to equilibrium.

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1 Introduction

The Neolithic migration of farmers in regions previously inhabited by hunter-gatherers has been studied for a long time [1, 2]. In particular the Lotka-Volterra type system

$$\begin{cases} F_t = d_F \Delta F + r_F F(1 - F + aH), \\ H_t = d_H \Delta H + r_H H(1 - H - bF), \end{cases} \tag{1}$$

has been studied by [3]. In this model, the populations of farmers F and hunter-gatherers H are assumed to diffuse freely and randomly by linear diffusion with constant diffusion rates d_F and d_H throughout the region. Recently, Eliaš, Kabir and Mimura [15] have proposed a new three component reaction-diffusion system

$$(\mathscr{P}^k) \begin{cases} F_{1,t} = F_1(1 - F_1 - F_2) + sF_1 H - k\left(p(F_1 + F_2)F_1 - (1 - p(F_1 + F_2))F_2\right), \\ F_{2,t} = d\Delta F_2 + F_2(1 - F_1 - F_2) + sF_2 H + k\left(p(F_1 + F_2)F_1 - (1 - p(F_1 + F_2))F_2\right), \\ H_t = \Delta H + rH(1 - H) - g(F_1 + F_2)H, \end{cases}$$

allowing to monitor the expanding farming population in terms of the sedentary and migrating farmers denoted by F_1 and F_2, respectively. In Problem $(\mathscr{P}^k)$, $p = p(F)$ is the probability density function which is included in the switching mechanism between the sedentary and migrating farmers and which depends on the total density of the farmers $F = F_1 + F_2$. We assume that p satisfies

$$\begin{cases} (i)\ p(0) = 0, \\ (ii)\ p(F) \text{ is increasing in } F, \\ (iii)\ \lim_{F \to \infty} p(F) = 1. \end{cases}$$

A simple example is given by $p(F) = F/(F + F_c)$, where F_c is the switching value of the conversion between F_1 and F_2; more precisely, the probabilities of remaining sedentary or migrating are both equal to 1/2 when $F = F_c$. Finally the parameter $k > 0$ is the rate of conversion between F_1 and F_2. In view of (i)-(iii), the model $(\mathscr{P}^k)$ implies that whenever the total density of farmers is low, the farmers prefer a sedentary lifestyle. On the other hand, if the total density of farmers is high, then some of the farmers start migrating and searching for new places favourable for sedentary life.

In this paper, we consider the special case when the rate of conversion k in Problem $(\mathscr{P}^k)$ tends to ∞. Formal calculations show and we will prove in a forthcoming article [14] that $(F_{k,1} + F_{k,2}, H_k)$ converges to (F, H) as $k \to \infty$, where the triple $(F_{k,1}, F_{k,2}, H_k)$ satisfies Problem $(\mathscr{P}^k)$ and (F, H) is a solution of the system

$$\begin{cases} F_t = d_F \Delta(p(F)F) + r_F F(1 - F) + sFH, \\ H_t = d_H \Delta H + r_H H(1 - H) - gFH. \end{cases} \tag{2}$$

Unlike in the system (1), the diffusion of farmers may degenerate if $p = 0$ in (cf. assumption (i)). In this model, the Neolithic dispersal of farming in Europe takes into account the population density pressure due to limited space and the advanced lifestyle resulting in farmer overcrowding.

More precisely, we study the nondimensionalised model

$$(\mathscr{P})\begin{cases} u_t = d_u\Delta\varphi(u) + r_u u(1-u+av) & \text{in } Q_T,\\ v_t = d_v\Delta v + r_v v(1-v-bu) & \text{in } Q_T,\\ \dfrac{\partial\varphi(u)}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0 & \text{on } \Sigma_T,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in\Omega, \end{cases}$$

where the dispersal of the hunting-gathering population v is assumed to be linear and the dispersal of the farming population u is modelled by using a possibly degenerate diffusion φ where $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ satisfies

$$(H_\varphi)\quad \begin{array}{l} \varphi\in C^3(\mathbb{R}_+)\cap C^1(\overline{\mathbb{R}_+}),\ \ \varphi(0)=\varphi'(0)=0,\ \ \varphi'(s)>0 \text{ for } s>0 \ \text{ and}\\ \varphi''(s)\geq 0 \text{ for } s\in[0,2C_a], \end{array}$$

where $C_a = 1+a$. We assume that u and v are defined on an open bounded domain $\Omega\subset\mathbb{R}^d$ with a smooth boundary and that the initial functions $u_0, v_0 \in C(\overline{\Omega})$ satisfy $0\leq u_0\leq C_a$ and $0\leq v_0\leq 1$. For $T>0$ we use the notation $Q_T=\Omega\times(0,T)$, $\Sigma_T=\partial\Omega\times(0,T)$ and ν denotes the outward normal at $x\in\partial\Omega$. In the model $(\mathscr{P})$, hunters are converted to farmers with the conversion rates a and b; r_u and r_v are the intrinsic per-capita growth rates and the carrying capacities of the habitat for farmers and hunter-gatherers are rescaled to 1. All the rates including the diffusion constants d_u and d_v are assumed to be positive constants. The system (2) is a special case of Problem $(\mathscr{P})$.

In this paper we prove the global in time existence and uniqueness of the solution of Problem $(\mathscr{P})$ and we study the large time behaviour of the solution as $t\to\infty$. In particular, depending on the value of b, we show that both farmers and hunters can coexist ($0<b<1$) or that hunters can become extinct ($b\geq 1$); in any case the population densities converge to constant steady states so that the populations homogeneously spread over the space domain.

Our main result is to prove that the convergence to equilibrium is exponential in the L^p topology for all $p\geq 1$. In other words we show that the constant equilibria of Problem $(\mathscr{P})$ are exponentially asymptotically stable in L^p for all $p\geq 1$ and we provide some explicit estimates for the convergence rates and constants. The method that we apply is inspired from the so-called entropy method, which measures the distance between the solution and the stationary state by means of a suitable, monotone in time Lyapunov (entropy or free energy) functional of the system. The idea is to establish functional inequalities between this Lyapunov functional, say, V, and the associated dissipation functional $\mathrm{d}V/\mathrm{d}t$. The entropy method has mainly been developed in the framework of scalar diffusion equations and the kinetic theory of the spatially homogeneous Boltzmann equation, see [4, 7, 23] and references therein. The method has also been used to obtain explicit rates for the exponential decay to equilibrium in the case of reaction-diffusion systems modelling reversible chemical reactions such as $2A_1\rightleftharpoons A_2$, $A_1+A_2\rightleftharpoons A_3$, $A_1+A_2\rightleftharpoons A_3+A_4$ and $A_1+A_2\rightleftharpoons A_3\rightleftharpoons A_1+A_4$ in [8, 9, 10, 16, 13]. The usual entropy functional used in the case of reversible reactions has the form

$$V(a_1,a_2,\ldots)=\sum_i\int_\Omega(a_i\log a_i - a_i+1)\,\mathrm{d}x \tag{3}$$

where a_i is the molar concentration of the chemical A_i and the summation goes through all the species of the reaction under consideration.

For Problem $(\mathscr{P})$, if $0 < b < 1$, let (u^*, v^*) be a strictly positive spatially homogeneous steady state solution of Problem $(\mathscr{P})$. The Lyapunov functional, which we will use, is given by

$$V(u,v) = \alpha \int_\Omega \Big(u - u^* - u^* \log \frac{u}{u^*} \Big) \, \mathrm{d}x + \beta \int_\Omega \Big(v - v^* - v^* \log \frac{v}{v^*} \Big) \, \mathrm{d}x. \tag{4}$$

In contrast, if $b \geq 1$, a steady state solution of Problem $(\mathscr{P})$ is given by $(1,0)$ and the functional

$$V(u,v) = \alpha \int_\Omega (u - 1 - \log u) \, \mathrm{d}x + \beta \int_\Omega v \, \mathrm{d}x \tag{5}$$

is a Lyapunov functional of Problem $(\mathscr{P})$.

There is an essential difference between the functionals (3) and (4), (5). In particular the functional (3) is well defined even if one or more species vanishes. On the other hand the functionals (4) and (5) blow up whenever u and/or v are not strictly positive in Ω. As it is expected for u to be zero in a part of the domain due to the finite time propagation property of solutions of porous medium equations starting from compactly supported initial data, we first need to show the eventual positivity of the solution (u,v) of Problem $(\mathscr{P})$ everywhere in Ω, which makes our approach different from the previous studies about reversible chemical reactions. The eventual positivity will be deduced from the uniform convergence of the solution (u,v) of Problem $(\mathscr{P})$ to the equilibrium in $(C(\overline{\Omega}))^2$. Indeed, having this uniform convergence, we can find a time, say $t_\mu > 0$, after which u and v are bounded below by a suitably small positive constant whenever the corresponding equilibrium is also strictly positive. For all the times $t \geq t_\mu$, the desired functional inequality between the Lyapunov functional and its time derivative has the form of $\mathrm{d}V/\mathrm{d}t \leq -CV$ for some positive constant C, which immediately implies the exponential convergence of the functional $V(u(t), v(t))$ to its steady state as $t \to \infty$. Then a Pinsker's-type inequality allows to find a lower bound for V in the sense of the L^2 distance between u and u^* and v and v^*, respectively, which in turn implies the exponential convergence of the solution orbits towards their steady state. This method works out in the cases that $0 < b < 1$ and $b > 1$. If $b = 1$, we can only prove an inequality of the form $\mathrm{d}V/\mathrm{d}t \leq -CV^2$, which then yields an algebraic convergence result.

The paper is organised as follows. In Section 2, we prove the existence and uniqueness of the solution of the degenerate parabolic problem $\mathscr{P}$ in the sense of Definition 2.1. Then, we will study the asymptotic behaviour of the solution of Problem $\mathscr{P}$ as time tends to infinity in Section 3. For the sake of completeness, Section 3 is concluded with remarks on the stabilisation for large time of a uniformly parabolic problem which corresponds to Problem $\mathscr{P}$. Namely, we will assume in Subsection 3.4 that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$(\widetilde{H}_\varphi) \quad \varphi \in C^2(\mathbb{R}_+) \text{ and } \varphi'(s) > 0 \text{ for } s \geq 0$$

and that u_0 and v_0 are bounded away from zero. Under the hypothesis $(\widetilde{H}_\varphi)$, Problem $\mathscr{P}$ is parabolic non-degenerate, and we may apply standard quasilinear theory to obtain the existence and uniqueness of classical solution, which is positive in $\Omega \times (0,\infty)$ for all nonnegative and compactly supported initial data.

2 Existence and uniqueness of the solution of Problem $(\mathscr{P})$

Let us consider the sequence of approximating problems

$$(\mathscr{P}^\varepsilon)\begin{cases} u_t = d_u\Delta\varphi_\varepsilon(u) + r_u u(1-u+av) & \text{in } Q_T,\\ v_t = d_v\Delta v + r_v v(1-v-bu) & \text{in } Q_T,\\ \dfrac{\partial\varphi_\varepsilon(u)}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0 & \text{on } \Sigma_T,\\ u(x,0) = u_0^\varepsilon(x),\ v(x,0) = v_0^\varepsilon(x), & x\in\Omega, \end{cases}$$

where $\varphi_\varepsilon(u) := \varphi(u+\varepsilon) - \varphi(\varepsilon)$. We remark that the function $\varphi_\varepsilon \in C^3([0,\infty))$ is such that

$$\varphi_\varepsilon(0) = 0 \ \text{ and } \ \varphi_\varepsilon'(s) > 0 \ \text{ for all } s \geq 0.$$

The initial functions u_0^ε and v_0^ε are smooth, i.e., $u_0^\varepsilon, v_0^\varepsilon \in C^\infty(\overline{\Omega})$, satisfy $0 \leq u_0^\varepsilon \leq 1+a$ and $0 \leq v_0^\varepsilon \leq 1$, and $u_0^\varepsilon \to u_0$, $v_0^\varepsilon \to v_0$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$.

Theorem 2.1. *There exists a unique classical solution $(u^\varepsilon, v^\varepsilon)$ of Problem $(\mathscr{P}^\varepsilon)$.*

Proof. The proof is based on Schauder's fixed point theorem (e.g., [12], Theorem. 5.1.11). Let us define the closed, convex and bounded set $\mathscr{K} = \{u \in C(\overline{Q}_T), 0 \leq u \leq 1+a \text{ in } \overline{Q}_T\}$.

Let $u^\varepsilon \in \mathscr{K}$. Then, by [20] (Proposition 7.3.2), there exists a unique solution $v^\varepsilon \in C^{2,1}(\overline{Q}_T)$ of the problem $(\mathscr{P}_v^\varepsilon)$,

$$(\mathscr{P}_v^\varepsilon)\begin{cases} v_t = d_v\Delta v + g(u^\varepsilon, v) & \text{in } Q_T,\\ \dfrac{\partial v}{\partial\nu} = 0 & \text{on } \Sigma_T,\\ v(x,0) = v_0^\varepsilon(x), & x\in\Omega, \end{cases}$$

where

$$g(u,v) = \begin{cases} 0 & \text{if } v<0,\\ r_v v(1-v-bu) & \text{if } 0\leq v\leq 1,\\ -r_v bu & \text{if } 1<v, \end{cases}$$

which is uniformly bounded, $0 \leq v^\varepsilon \leq 1$ in $\overline{Q}_T$. Indeed, we define

$$\mathscr{L}_v(w) = w_t - d_v\Delta w - g(u^\varepsilon, w).$$

The boundedness of v^ε follows from the comparison principle and the fact that $\mathscr{L}_v(0) \leq 0$ and $\mathscr{L}_v(1) \geq 0$ whenever $u^\varepsilon \geq 0$.

If v^ε is a solution of Problem $(\mathscr{P}_v^\varepsilon)$, then we consider the uniformly parabolic problem

$$(\mathscr{P}_u^\varepsilon)\begin{cases} u_t = d_u\Delta\varphi_\varepsilon(u) + f(u, v^\varepsilon) & \text{in } Q_T,\\ \dfrac{\partial\varphi_\varepsilon(u)}{\partial\nu} = 0 & \text{on } \Sigma_T,\\ u(x,0) = u_0^\varepsilon(x) & x\in\Omega, \end{cases}$$

where

$$f(u,v) = \begin{cases} 0 & \text{if } u<0,\\ r_u u(1-u+av) & \text{if } 0\leq u\leq 1+a,\\ -r_u a(1+a)(1-v) & \text{if } 1+a<u. \end{cases}$$

It follows from [19] (Chap. V, Theorem 7.4) that Problem $(\mathscr{P}_u^\varepsilon)$ possesses a unique classical solution $\widehat{u}^\varepsilon \in C^{2+\alpha,(2+\alpha)/2}(\overline{Q}_T)$ for all $\alpha \in (0,1)$. Moreover, $\widehat{u}^\varepsilon$ is uniformly bounded in $\overline{Q}_T$, namely $0 \le \widehat{u}^\varepsilon \le C_a := 1+a$ for each $(x,t) \in \overline{Q}_T$. Indeed, let us define

$$\mathscr{L}_u(w) = w_t - d_u \Delta \varphi_\varepsilon(w) - f(w, v^\varepsilon).$$

Then, $\mathscr{L}_u(0) \le 0$ and $\mathscr{L}_u(C_a) \ge 0$ and the uniform boundedness follows from the standard comparison principle.

We consider the map $\mathscr{F} : \mathscr{K} \to \mathscr{K}$ such that $u^\varepsilon \mapsto v^\varepsilon \mapsto \widehat{u}^\varepsilon =: \mathscr{F}(u^\varepsilon)$. In order to apply Schauder's fixed point theorem we have to show that the map $\mathscr{F}$ is compact and continuous from $\mathscr{K}$ into $C(\overline{Q}_T)$.

(i) compactness of $\mathscr{F}$: If $\{u_j^\varepsilon\}_{j=1}^\infty$ is a sequence of functions in $\mathscr{K}$, then it follows from the regularity of all $\widehat{u}_j^\varepsilon$ and the compactness of the embedding $C^{2+\alpha,(2+\alpha)/2}(\overline{Q}_T) \subset C(\overline{Q}_T)$ that the sequence $\{\mathscr{F}(u_j^\varepsilon)\}_{j=1}^\infty$ is relatively compact in $C(\overline{Q}_T)$.

(ii) continuity of $\mathscr{F}$: For any two solutions v_1^ε and v_2^ε of Problem $(\mathscr{P}_v^\varepsilon)$ which correspond to the functions u_1^ε and u_2^ε from $\mathscr{K}$ and the initial functions $v_{1,0}^\varepsilon$ and $v_{2,0}^\varepsilon$, respectively, and such that $0 \le v_1^\varepsilon, v_2^\varepsilon \le 1$, we deduce that

$$\begin{aligned}\int_\Omega |g(u_1^\varepsilon, v_1^\varepsilon) - g(u_2^\varepsilon, v_2^\varepsilon)|\,\mathrm{d}x &= r_v \int_\Omega |v_1^\varepsilon(1 - v_1^\varepsilon - bu_1^\varepsilon) - v_2^\varepsilon(1 - v_2^\varepsilon - bu_2^\varepsilon)|\,\mathrm{d}x \\ &= r_v \int_\Omega \left|v_1^\varepsilon - v_2^\varepsilon - ((v_1^\varepsilon)^2 - (v_2^\varepsilon)^2) + bv_2^\varepsilon(u_2^\varepsilon - u_1^\varepsilon) + bu_1^\varepsilon(v_2^\varepsilon - v_1^\varepsilon)\right|\mathrm{d}x \\ &\le r_v(3 + bC_a) \int_\Omega |v_1^\varepsilon - v_2^\varepsilon|\,\mathrm{d}x + br_v \int_\Omega |u_1^\varepsilon - u_2^\varepsilon|\,\mathrm{d}x.\end{aligned}$$

With this estimate at hand, the stability property in [5], Corollary 11, namely,

$$\|v_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{L^1(\Omega)} \le \left\|v_{1,0}^\varepsilon - v_{2,0}^\varepsilon\right\|_{L^1(\Omega)} + \int_0^t \|g(u_1^\varepsilon, v_1^\varepsilon) - g(u_2^\varepsilon, v_2^\varepsilon)\|_{L^1(\Omega)}\,\mathrm{d}s,$$

implies that

$$\|v_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{L^1(\Omega)} \le \left\|v_{1,0}^\varepsilon - v_{2,0}^\varepsilon\right\|_{L^1(\Omega)} + C\left(\int_0^t \|v_1^\varepsilon - v_2^\varepsilon\|_{L^1(\Omega)}\,\mathrm{d}s + \int_0^t \|u_1^\varepsilon - u_2^\varepsilon\|_{L^1(\Omega)}\,\mathrm{d}s\right) \quad (6)$$

for some positive constant C. By using Gronwall's inequality we deduce that[1]

$$\|v_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{L^1(\Omega)} \le e^{Ct}\left(\left\|v_{1,0}^\varepsilon - v_{2,0}^\varepsilon\right\|_{L^1(\Omega)} + C\int_0^t \|u_1^\varepsilon - u_2^\varepsilon\|_{L^1(\Omega)}\,\mathrm{d}s\right)$$

[1] Let $y(t) = \left\|v_1^\varepsilon(t) - v_2^\varepsilon(t)\right\|_{L^1(\Omega)}$, $y(0) = \|v_{1,0}^\varepsilon - v_{2,0}^\varepsilon\|_{L^1(\Omega)}$, $b(t) = \left\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\right\|_{L^1(\Omega)}$ and $C > 0$. Then, (6) can be rewritten for $t \ge 0$

$$y(t) \le y(0) + C\int_0^t (y(s) + b(s))\,\mathrm{d}s.$$

Let us denote

$$x(t) = y(0) + C\int_0^t (y(s) + b(s))\,\mathrm{d}s.$$

for $0 < t \leq T$, so that

$$\int_0^T \|v_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{L^1(\Omega)}\,\mathrm{d}t \leq \frac{1}{C} e^{CT} \left(\left\| v_{1,0}^\varepsilon - v_{2,0}^\varepsilon \right\|_{L^1(\Omega)} + C \int_0^T \|u_1^\varepsilon - u_2^\varepsilon\|_{L^1(\Omega)}\,\mathrm{d}s \right). \tag{7}$$

Similarly, any two solutions $\widehat{u}_1^\varepsilon$ and $\widehat{u}_2^\varepsilon$ of Problem $(\mathscr{P}_u^\varepsilon)$ that correspond to v_1^ε, v_2^ε and $u_{1,0}^\varepsilon$, $u_{2,0}^\varepsilon$ satisfy

$$\|\widehat{u}_1^\varepsilon(t) - \widehat{u}_2^\varepsilon(t)\|_{L^1(\Omega)} \leq \left\| u_{1,0}^\varepsilon - u_{2,0}^\varepsilon \right\|_{L^1(\Omega)} + C \left(\int_0^t \|\widehat{u}_1^\varepsilon - \widehat{u}_2^\varepsilon\|_{L^1(\Omega)}\,\mathrm{d}s + \int_0^t \|v_1^\varepsilon - v_2^\varepsilon\|_{L^1(\Omega)}\,\mathrm{d}s \right) \tag{8}$$

where C is a constant. Therefore, we deduce the inequality

$$\int_0^T \|\widehat{u}_1^\varepsilon(t) - \widehat{u}_2^\varepsilon(t)\|_{L^1(\Omega)}\,\mathrm{d}t \leq \frac{1}{C} e^{CT} \left(\left\| u_{1,0}^\varepsilon - u_{2,0}^\varepsilon \right\|_{L^1(\Omega)} + C \int_0^T \|v_1^\varepsilon - v_2^\varepsilon\|_{L^1(\Omega)}\,\mathrm{d}s \right). \tag{9}$$

Let us consider a convergent sequence $\{u_j^\varepsilon\}_{j=1}^\infty$ in $\mathscr{K}$ and denote its limit by u^ε, i.e., $u_j^\varepsilon \to u^\varepsilon$ in $\mathscr{K}$ as $j \to \infty$. Assume also that the initial functions satisfy $v_{j,0}^\varepsilon = v_0^\varepsilon$ and $\widehat{u}_{j,0}^\varepsilon = u_0^\varepsilon$. Then,

$$\begin{aligned} \int_0^T \left\| \mathscr{F}(u_j^\varepsilon) - \mathscr{F}(u^\varepsilon) \right\|_{L^1(\Omega)} \mathrm{d}s &= \int_0^T \left\| \widehat{u}_j^\varepsilon - \widehat{u}^\varepsilon \right\|_{L^1(\Omega)} \mathrm{d}s \\ &\leq e^{CT} \int_0^T \left\| v_j^\varepsilon - v^\varepsilon \right\|_{L^1(\Omega)} \mathrm{d}s \\ &\leq e^{2CT} \int_0^T \left\| u_j^\varepsilon - u^\varepsilon \right\|_{L^1(\Omega)} \mathrm{d}s \end{aligned} \tag{10}$$

by (7) and (9). Hence, it follows from (10) that $\mathscr{F}$ is continuous in the $L^1(Q_T)$ norm and $\mathscr{F}(u_j^\varepsilon) \to \mathscr{F}(u^\varepsilon)$ in $L^1(Q_T)$ as $j \to \infty$. Since $\mathscr{F}$ is a compact map from $\mathscr{K}$ to $C(\overline{Q}_T)$ we deduce that $\mathscr{F}(u_j^\varepsilon) \to \mathscr{F}(u^\varepsilon)$ in $C(\overline{Q}_T)$ as $j \to \infty$.

Then, $y(t) \leq x(t)$ for $t > 0$, $x(0) = y(0)$ and,

$$x'(t) = C(y(t) + b(t) - y(0) - b(0)).$$

Since $y(t) \leq x(t)$ and $y(0) \geq 0$ and $b(0) \geq 0$, then we deduce

$$x'(t) \leq C(x(t) + b(t)),$$

and, after multiplication by e^{-Ct},

$$(x(t)e^{-Ct})' \leq Cb(t)e^{-Ct}.$$

Integration over $(0,t)$ then gives

$$x(t)e^{-Ct} - x(0) \leq C \int_0^t b(s)e^{-Cs}\,\mathrm{d}s.$$

Since $y(t) \leq x(t)$ and $x(0) = y(0)$, we deduce that

$$y(t) \leq x(t) \leq e^{Ct} \left(y(0) + C \int_0^t b(s)e^{-Cs}\,\mathrm{d}s \right) \leq e^{Ct} \left(y(0) + C \int_0^t b(s)\,\mathrm{d}s \right).$$

Please note that a similar integral form of the Gronwall inequality can be found in [21] on p. 25.

Therefore, $\mathscr{F} : u^\varepsilon \mapsto \widehat{u}^\varepsilon$ is continuous and compact for the $C(\overline{Q}_T)$ topology from the closed, convex, bounded set $\mathscr{K}$ into itself. We deduce from Schauder's fixed point theorem that there exists a function $u^\varepsilon \in \mathscr{K}$ such that $\mathscr{F}(u^\varepsilon) = u^\varepsilon$. This proves the existence of a solution of Problem $(\mathscr{P}^\varepsilon)$.

The uniqueness of the solution follows from the stability properties (6) and (8): for $t \in [0,T]$,

$$\begin{aligned}
\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\|_{L^1(\Omega)} + \|v_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{L^1(\Omega)} \\
\leq \left\|u_{1,0}^\varepsilon - u_{2,0}^\varepsilon\right\|_{L^1(\Omega)} + \left\|v_{1,0}^\varepsilon - v_{2,0}^\varepsilon\right\|_{L^1(\Omega)} \\
+ C\int_0^t \Big(\|u_1^\varepsilon(s) - u_2^\varepsilon(s)\|_{L^1(\Omega)} + \|v_1^\varepsilon(s) - v_2^\varepsilon(s)\|_{L^1(\Omega)}\Big)\, \mathrm{d}s.
\end{aligned}$$

Thus, by Gronwall's inequality we obtain

$$\begin{aligned}
\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\|_{L^1(\Omega)} + \|v_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{L^1(\Omega)} \\
\leq e^{Ct}\Big(\left\|u_{1,0}^\varepsilon - u_{2,0}^\varepsilon\right\|_{L^1(\Omega)} + \left\|v_{1,0}^\varepsilon - v_{2,0}^\varepsilon\right\|_{L^1(\Omega)}\Big)
\end{aligned}$$

for each $t \in [0,T]$. If we set $u_{1,0}^\varepsilon = u_{2,0}^\varepsilon$ and $v_{1,0}^\varepsilon = v_{2,0}^\varepsilon$ we deduce that $u_1^\varepsilon = u_2^\varepsilon$ and $v_1^\varepsilon = v_2^\varepsilon$ for each $t \in [0,T]$ and a.e. in Ω. □

Next we return to the study of Problem $(\mathscr{P})$.

Definition 2.1. We say that a pair (u,v) is a weak solution to Problem $(\mathscr{P})$ if

i) $u,v \in C(\overline{Q}_T)$ and

ii) for any $\zeta \in C^{2,1}(\overline{Q}_T)$, $\partial_\nu \zeta = 0$ on Σ_T and for each $t \in [0,T]$

$$\int_\Omega u(t)\zeta(t)\,\mathrm{d}x = \int_\Omega u_0\zeta(0)\,\mathrm{d}x + \int_0^t\int_\Omega (d_u\varphi(u)\Delta\zeta + r_u u(1-u+av)\zeta + u\zeta_t)\,\mathrm{d}x\,\mathrm{d}s \tag{11}$$

and

$$\int_\Omega v(t)\zeta(t)\,\mathrm{d}x = \int_\Omega v_0\zeta(0)\,\mathrm{d}x + \int_0^t\int_\Omega (d_v v\Delta\zeta + r_v v(1-v-bu)\zeta + v\zeta_t)\,\mathrm{d}x\,\mathrm{d}s. \tag{12}$$

Theorem 2.2. *Problem $(\mathscr{P})$ admits a unique weak solution (u,v) such that*

$$0 \leq u \leq 1+a \quad \text{and} \quad 0 \leq v \leq 1. \tag{13}$$

in $\overline{\Omega} \times [0,\infty)$.

Proof. We first prove the existence of the solution. To that purpose, we will show that the sequence $\{(u^\varepsilon, v^\varepsilon)\}$ converges uniformly to a limit (u,v) as $\varepsilon \to 0$, where (u,v) turns out to be the unique weak solution of Problem $(\mathscr{P})$. In fact, it is handy to set $U^\varepsilon = \varphi_\varepsilon(u^\varepsilon)$ and to prove the uniform convergence of $\{U^\varepsilon\}$ in $C(\overline{Q}_T)$. We remark that since $0 \leq u^\varepsilon \leq 1+a$ in $\overline{Q}_T$ and φ is increasing on $[0,\infty)$, we have that $0 \leq \varphi_\varepsilon(u^\varepsilon) = \varphi(u^\varepsilon+\varepsilon) - \varphi(\varepsilon) \leq \varphi(1+a+\varepsilon) \leq \varphi(2C_a)$, where $C_a = 1+a$, as we may assume without loss of generality that $\varepsilon < C_a$. Hence $0 \leq U^\varepsilon \leq \varphi(2+a)$ in $\overline{Q}_T$.

In view of (H_φ) and the definition of φ_ε, the function φ_ε is a convex strictly increasing function from $[0,2C_a]$ into $[0,\varphi_\varepsilon(2C_a)]$. Hence, we can define $\beta_\varepsilon = \varphi_\varepsilon^{-1}$ which is a concave strictly increasing function on the interval $[0,\varphi_\varepsilon(2C_a)]$. Since φ is convex on $[0,2C_a]$, the function $\varepsilon \mapsto \varphi_\varepsilon(s)$ is nondecreasing at each $s \in [0,2C_a]$. Indeed, define $f(\varepsilon) = \varphi_\varepsilon(s)$; then $f'(\varepsilon) = \varphi'(s+\varepsilon) - \varphi'(\varepsilon) \geq 0$ thanks to the convexity of φ. We deduce that $\varphi(s) \leq \varphi_\varepsilon(s)$ for all $s \in [0,2C_a]$ and, in particular, $[0,\varphi(2C_a)] \subset [0,\varphi_\varepsilon(2C_a)]$ for all $0 \leq \varepsilon \leq C_a$. Since $\varphi_\varepsilon(u^\varepsilon) \in [0,\varphi(2C_a)]$ for all $\varepsilon \leq C_a$, we can restrict the definition of β_ε to the interval $[0,\varphi(2C_a)]$.

Next, we define $\beta(s) = \varphi^{-1}(s)$ for $s \in [0,\varphi(2C_a)]$. Then,

$$\beta_\varepsilon \to \beta \quad \text{uniformly on } [0,\varphi(2C_a)] \text{ as } \varepsilon \to 0.$$

Indeed, the function $\varepsilon \mapsto \beta_\varepsilon(s)$ is nonincreasing for each $s \in [0,\varphi(2C_a)]$. Thus, the sequence $\{\beta_\varepsilon\}_{\varepsilon>0}$ is a monotone sequence of continuous functions tending pointwise to the continuous function β on the compact set $[0,\varphi(2C_a)]$ as $\varepsilon \to 0$. We deduce the uniform convergence from Dini's theorem.

By setting $U^\varepsilon = \varphi_\varepsilon(u^\varepsilon)$ we obtain the equation for U^ε,

$$\begin{cases} \dfrac{\partial}{\partial t}\beta_\varepsilon(U^\varepsilon) = d_u \Delta U^\varepsilon + f(\beta_\varepsilon(U^\varepsilon), v^\varepsilon) & \text{in } Q_T, \\ \dfrac{\partial}{\partial \nu} U^\varepsilon = 0 & \text{on } \Sigma_T, \\ U^\varepsilon(x,0) = \varphi_\varepsilon(u_0^\varepsilon(x)), & x \in \Omega, \end{cases}$$

where $f(\beta_\varepsilon(U^\varepsilon), v^\varepsilon) = r_u \beta_\varepsilon(U^\varepsilon)(1 - \beta_\varepsilon(U^\varepsilon) + a v^\varepsilon)$ is bounded uniformly in ε. Multiplying the equation for U^ε by U^ε and integrating over Ω gives

$$\int_\Omega \frac{\partial \beta_\varepsilon(U^\varepsilon)}{\partial t} U^\varepsilon \,\mathrm{d}x + d_u \int_\Omega |\nabla U^\varepsilon|^2 \,\mathrm{d}x = \int_\Omega f(\beta_\varepsilon(U^\varepsilon), v^\varepsilon) U^\varepsilon \,\mathrm{d}x.$$

Integration in time then implies that

$$\int_0^T \int_\Omega \frac{\partial \beta_\varepsilon(U^\varepsilon)}{\partial t} U^\varepsilon \,\mathrm{d}x\,\mathrm{d}t + d_u \int_0^T \int_\Omega |\nabla U^\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C_1, \tag{14}$$

where the positive constant C_1 does not depend on ε.

We set $F_\varepsilon(s) = \int_0^s \beta_\varepsilon'(\zeta)\zeta \,\mathrm{d}\zeta$. Since $\varphi_\varepsilon'(s) \geq 0$ for $s \geq 0$, then also $\beta_\varepsilon'(s) \geq 0$ and $F_\varepsilon(s) \geq 0$ for all $s \geq 0$. We obtain

$$\int_0^T \int_\Omega \frac{\partial \beta_\varepsilon(U^\varepsilon)}{\partial t} U^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \frac{\partial F_\varepsilon(U^\varepsilon)}{\partial t} \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega F_\varepsilon(U^\varepsilon(T)) \,\mathrm{d}x - \int_\Omega F_\varepsilon(U^\varepsilon(0)) \,\mathrm{d}x \tag{15}$$

where

$$\begin{aligned} \int_\Omega F_\varepsilon(U^\varepsilon(0)) \,\mathrm{d}x &= \int_\Omega F_\varepsilon(\varphi_\varepsilon(u_0^\varepsilon)) \,\mathrm{d}x \\ &= \int_\Omega \int_0^{\varphi_\varepsilon(u_0^\varepsilon)} \beta_\varepsilon'(\zeta)\zeta \,\mathrm{d}\zeta \,\mathrm{d}x \\ &\leq \int_\Omega \varphi_\varepsilon(u_0^\varepsilon) \int_0^{\varphi_\varepsilon(u_0^\varepsilon)} \beta_\varepsilon'(\zeta) \,\mathrm{d}\zeta \,\mathrm{d}x \\ &= \int_\Omega \varphi_\varepsilon(u_0^\varepsilon)(\beta_\varepsilon(\varphi_\varepsilon(u_0^\varepsilon)) - \beta_\varepsilon(0)) \,\mathrm{d}x \\ &= \int_\Omega \varphi_\varepsilon(u_0^\varepsilon) u_0^\varepsilon \,\mathrm{d}x \\ &\leq C_2 \end{aligned} \tag{16}$$

where $C_2 = (1+a)\varphi(2+a)|\Omega|$ since $0 \le u_0^\varepsilon \le 1+a$ and $0 \le \varphi_\varepsilon(u_0^\varepsilon) \le \varphi(2+a)$. By substituting (15) and (16) into (14) we obtain

$$d_u \int_0^T \int_\Omega |\nabla U^\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \le C_1 + C_2.$$

This implies that $\{U^\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2(0,T;H^1(\Omega))$ uniformly in ε.

Since $u_0^\varepsilon \to u_0$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$, there exists a positive function ω such that $\omega(s) \to 0$ as $s \to 0$ and such that for all $0 < \varepsilon < \varepsilon_0$ we have that

$$\left| u_0^\varepsilon(x) - u_0^\varepsilon(x') \right| \le \omega(x-x') \quad \text{for all } x, x' \in \overline{\Omega},$$

so that also

$$\left| u_0(x) - u_0(x') \right| \le \omega(x-x') \quad \text{for all } x, x' \in \overline{\Omega}.$$

By the results of DiBenedetto [11] (Theorem 6.2 and the following corollary) we deduce that $\{U^\varepsilon\}_{\varepsilon>0}$ is equicontinuous in $\overline{Q}_T$ and thus relatively compact in $C(\overline{Q}_T)$. Thus, there exists $U \in C(\overline{Q}_T)$ and a subsequence of $\{U^\varepsilon\}$ (denoted again by $\{U^\varepsilon\}$) such that $U^\varepsilon \to U$ uniformly in $C(\overline{Q}_T)$ as $\varepsilon \to 0$.

Since $\beta_\varepsilon(s) \to \beta(s)$ uniformly on $[0, \varphi(2C_a)]$ as $\varepsilon \to 0$ and $\beta(U^\varepsilon) \to \beta(U)$ uniformly in $C(\overline{Q}_T)$ as $\varepsilon \to 0$, we deduce from the inequality

$$|u^\varepsilon - \beta(U)| = |\beta_\varepsilon(U^\varepsilon) - \beta(U)| \le |\beta_\varepsilon(U^\varepsilon) - \beta(U^\varepsilon)| + |(\beta(U^\varepsilon) - \beta(U)|$$

that

$$u^\varepsilon \to u \quad \text{uniformly in } C(\overline{Q}_T) \text{ as } \varepsilon \to 0, \tag{17}$$

where we set $u = \beta(U)$ (i.e., $U = \varphi(u)$).

A similar reasoning permits to show that

$$v^\varepsilon \to v \quad \text{uniformly in } C(\overline{Q}_T) \text{ as } \varepsilon \to 0. \tag{18}$$

Let $\zeta \in C^{2,1}(\overline{Q}_T)$ be such that $\partial_\nu \zeta = 0$ for $(x,t) \in \Sigma_T$. Using the uniform convergence properties (17) and (18) and a similar property for $\varphi_\varepsilon(u^\varepsilon)$, Lebesgue's dominated convergence theorem and the fact that $u_0^\varepsilon \to u_0$ and $v_0^\varepsilon \to v_0$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$ allows us to pass to the limit $\varepsilon \to 0$ in the weak formulation

$$\int_\Omega u^\varepsilon(t)\zeta(t)\,\mathrm{d}x = \int_\Omega u_0^\varepsilon \zeta(0)\,\mathrm{d}x + d_u \int_0^t \int_\Omega \varphi_\varepsilon(u^\varepsilon)\Delta\zeta \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega u^\varepsilon \zeta_t \,\mathrm{d}x\,\mathrm{d}s$$
$$+ r_u \int_0^t \int_\Omega u^\varepsilon(1 - u^\varepsilon + a v^\varepsilon)\zeta \,\mathrm{d}x\,\mathrm{d}s$$

and

$$\int_\Omega v^\varepsilon(t)\zeta(t)\,\mathrm{d}x = \int_\Omega v_0^\varepsilon \zeta(0)\,\mathrm{d}x + d_v \int_0^t \int_\Omega v^\varepsilon \Delta\zeta \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega v^\varepsilon \zeta_t \,\mathrm{d}x\,\mathrm{d}s$$
$$+ r_v \int_0^t \int_\Omega v^\varepsilon(1 - v^\varepsilon - b u^\varepsilon)\zeta \,\mathrm{d}x\,\mathrm{d}s$$

to obtain (11) and (12). Therefore (u,v) is a weak solution to Problem $(\mathscr{P})$.

We obtain the uniform bounds (13) as a consequence of the uniform bounds $0 \le u^\varepsilon \le 1+a$ and $0 \le v^\varepsilon \le 1$ for each $\varepsilon > 0$ and the uniform convergence properties (17) and (18).

The proof of the uniqueness of the solution is similar to the proof of the uniqueness in Theorem 2.1, since the stability properties (6) and (8) hold also for the solution of Problem $(\mathscr{P})$. □

3 Convergence to equilibrium as $t \to \infty$

We remark that the system $(\overline{\mathscr{P}})$ of ordinary differential equations that corresponds to $(\mathscr{P})$ admits several steady states in the positive quadrant $\mathbb{R}_+^2 = \{(r,s) : r \geq 0, s \geq 0\}$ depending on the value of b. If $0 < b < 1$, then the system $(\overline{\mathscr{P}})$ possesses four equilibria $(u_\infty, v_\infty) \in \{(0,0),(1,0),(0,1),(u^*,v^*)\}$ where $u^* = (1+a)/(1+ab) > 0$ and $v^* = (1-b)/(1+ab) > 0$. Stability analysis gives that the only stable equilibrium is $(u_\infty, v_\infty) = (u^*, v^*)$. In the case when $b \geq 1$, the set of equilibria is $\{(0,0),(1,0),(0,1)\}$, since $v^* < 0$ for $b > 1$. From these, $(u_\infty, v_\infty) = (1,0)$ is the only stable steady state of the system $(\overline{\mathscr{P}})$. We will show below that the equilibrium state

$$\begin{cases} (u^*,v^*) \text{ for } 0 < b < 1, \\ \quad (1,0) \text{ for } b \geq 1, \end{cases}$$

is the globally stable steady state solution of Problem $(\mathscr{P})$.

For later reference we define

$$\mathscr{L}_u(s) = s_t - d_u \Delta\varphi(s) - r_u s(1 - s + av), \tag{19}$$

$$\mathscr{L}_v(s) = s_t - d_v \Delta s - r_v s(1 - s - bu). \tag{20}$$

3.1 Positivity of the solution after some time

We start by proving that the solution (u,v) of Problem $(\mathscr{P})$ becomes positive in Ω in a finite time. First, we consider the solution of an initial value problem for a scalar nonlinear Fisher-KPP equation.

Lemma 3.1. *Let α, β be positive constants and let $w_0 \in C(\overline{\Omega})$ be a nonnegative function such that $\int_\Omega w_0(x)\,\mathrm{d}x > 0$. Then, the weak solution w of the Fisher-KPP equation with degenerate diffusion*

$$\begin{cases} w_t = d\Delta\varphi(w) + \alpha w(\beta - w) \quad \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial \varphi(w)}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ w(x,0) = w_0(x), \quad x \in \Omega, \end{cases} \tag{21}$$

converges exponentially fast to β in $C(\overline{\Omega})$ as $t \to \infty$, where the convergence rate only depends on α and β.

Proof. We remark that the functional $\mathscr{L}_w(s) = s_t - d\Delta\varphi(s) - \alpha s(\beta - s)$ associated with Problem (21) satisfies $\mathscr{L}_w(0) \leq 0$ and $\mathscr{L}_w(C) \geq 0$ for $C = \max\{\|w_0\|_{L^\infty}, \beta\}$. Hence we deduce from the weak maximum principle that $0 \leq w \leq \max\{\|w_0\|_{L^\infty}, \beta\}$ in $\overline{\Omega} \times [0,\infty)$. Let z be the unique solution of the problem

$$\begin{cases} z_t = d\Delta\varphi(z) \quad \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial \varphi(z)}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ z(x,0) = z_0(x) = \min\limits_{x\in\Omega}\{w_0(x), \beta\}, \quad x \in \Omega. \end{cases}$$

We deduce from the weak maximum principle that $0 \le z \le \beta$ in $\Omega \times (0,\infty)$. Moreover, since $\mathscr{L}_w(z) = -\alpha z(\beta - z) \le 0$ and $z_0(x) \le w_0(x)$ in Ω, it follows that $z \le w$ in $\overline{\Omega} \times [0,\infty)$ by the comparison principle. We set

$$a(z_0) = \frac{1}{|\Omega|} \int_\Omega z_0(x)\,\mathrm{d}x > 0.$$

Then, by Theorem 20.16 in [22],

$$\lim_{t\to\infty} |z(x,t) - a(z_0)| = 0$$

uniformly in $\overline{\Omega}$. Hence, for each $0 < \mu < a(z_0)$ there exists $t_\mu > 0$ such that $w(x,t) \ge z(x,t) \ge \mu > 0$ in $\overline{\Omega} \times [t_\mu,\infty)$. For further use we fix the pair (μ, t_μ) with $0 < \mu < a(z_0)$.

We remark that

$$\underline{w} \le w \le \overline{w} \quad \text{in } \overline{\Omega} \times [t_\mu,\infty), \tag{22}$$

where $\underline{w}$ and $\overline{w}$ are the solutions of, respectively,

$$\begin{cases} \underline{w}_t = \alpha \underline{w}(\beta - \underline{w}) & \text{in } \Omega \times (t_\mu,\infty), \\ \underline{w}(x,t_\mu) = \min\{\mu, \beta/2\}, & x \in \Omega, \end{cases}$$

and

$$\begin{cases} \overline{w}_t = \alpha \overline{w}(\beta - \overline{w}) & \text{in } \Omega \times (t_\mu,\infty), \\ \overline{w}(x,t_\mu) = \max\{\|w_0\|_{L^\infty(\Omega)}, 2\beta\}, & x \in \Omega. \end{cases}$$

Indeed, both $\underline{w}$ and $\overline{w}$ satisfy $\mathscr{L}_w(\underline{w}) = \mathscr{L}_w(\overline{w}) = 0$ and

$$\underline{w}(\cdot,t_\mu) \le \mu \le w(\cdot,t_\mu) \le \max\{\|w_0\|_{L^\infty(\Omega)}, \beta\} \le \overline{w}(\cdot,t_\mu)$$

for the data at time t_μ in $\overline{\Omega}$. Thus, (22) follows from the comparison principle.

Since the solution of the ODE equation $n' = \alpha n(\beta - n)$ with $n(t_\mu) = n_0$,

$$n(t) = \frac{\beta}{1 + \gamma e^{-\alpha\beta(t - t_\mu)}}, \quad \gamma = \frac{\beta - n_0}{n_0},$$

converges to β exponentially fast as $t \to \infty$, then both $\underline{w}(t)$ and $\overline{w}(t)$ converge to β as $t \to \infty$, and as a consequence of (22), $w(t) \to \beta$ in $C(\overline{\Omega})$ exponentially fast as $t \to \infty$ for all $\alpha, \beta > 0$. □

Remark 3.1. We can prove in a similar way that the solution of the standard Fisher-KPP equation with linear diffusion (set $\varphi(w) = w$ in (21)) converges exponentially fast to β in $C(\overline{\Omega})$ as $t \to \infty$ for arbitrary $\alpha, \beta > 0$.

Corollary 3.2. *There exists $t_* > 0$ such that u and v are strictly positive in $\overline{\Omega} \times [t_*,\infty)$.*

Proof. Let $\underline{u}$ be the nonnegative solution of the problem

$$\begin{cases} \underline{u}_t = d_u \Delta\varphi(\underline{u}) + r_u \underline{u}(1 - \underline{u}) & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial \varphi(\underline{u})}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \underline{u}(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where u_0 is such that $0 \le u_0(x) \le 1$ in Ω. Since the functional $\mathscr{L}_u$ defined by (19) satisfies $\mathscr{L}_u(\underline{u}) \le -r_u a v \underline{u} \le 0$ for $v \ge 0$, then by the standard comparison principle we deduce that $u \ge \underline{u}$ in $\overline{\Omega} \times [0,\infty)$. Since by Lemma 3.1, $\underline{u}(t) \to 1$ in $C(\overline{\Omega})$ as $t \to \infty$, there exists $t_* > 0$ such that for all $t \ge t_*$ we have $u \ge \underline{u} > 0$ in $\overline{\Omega} \times [t_*,\infty)$.

Since the problem for v is uniformly parabolic, then $v(x,t) > 0$ in $\overline{\Omega} \times (0,\infty)$ by the strong maximum principle. □

3.2 Large time behaviour in the case that $0 < b < 1$

Lemma 3.3. *Let $0 < b < 1$, then the solution (u,v) of Problem ($\mathscr{P}$) converges to $(u^*,v^*) = ((1+a)/(1+ab),(1-b)/(1+ab))$ in $[C(\overline{\Omega})]^2$ as $t \to \infty$.*

Proof. We remark that the steady state solution (u^*,v^*) of Problem ($\mathscr{P}$) satisfies

$$1 = u^* - av^* \quad \text{and} \quad 1 = v^* + bu^*. \tag{23}$$

By Corollary 3.2 there exists a time t_* such that u and v are strictly positive in $\overline{\Omega} \times [t_*,\infty)$. Let us consider for $t \ge t_*$ the functional $V(u,v) : \mathbb{R}_+^2 \to \mathbb{R}_+$ defined by

$$V(u,v) = \alpha \int_\Omega (u - u^* - u^* \log \frac{u}{u^*})\,\mathrm{d}x + \beta \int_\Omega (v - v^* - v^* \log \frac{v}{v^*})\,\mathrm{d}x, \tag{24}$$

for some constants α, β still to be determined. Then V is positive for $(u,v) \neq (u^*,v^*)$ (see (33) below) and satisfies $V(u^*,v^*) = 0$. Differentiation of $V(u,v)$ in time gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(u(t),v(t)) = {} & \alpha \int_\Omega \frac{u-u^*}{u} u_t\,\mathrm{d}x + \beta \int_\Omega \frac{v-v^*}{v} v_t\,\mathrm{d}x \\ = \alpha & \int_\Omega \frac{u-u^*}{u}\left(d_u \Delta\varphi(u) + r_u u(1-u+av)\right)\mathrm{d}x + \beta \int_\Omega \frac{v-v^*}{v}\left(d_v \Delta v + r_v v(1-v-bu)\right)\mathrm{d}x\end{aligned}$$

where integration by parts yields

$$\alpha d_u \int_\Omega \frac{u-u^*}{u}\Delta\varphi(u)\,\mathrm{d}x = -\alpha d_u \int_\Omega \nabla\left(1 - \frac{u^*}{u}\right)\cdot\nabla\varphi(u)\,\mathrm{d}x = -\alpha d_u u^* \int_\Omega \varphi'(u)\frac{|\nabla u|^2}{u^2}\,\mathrm{d}x$$

and

$$\beta d_v \int_\Omega \frac{v-v^*}{v}\Delta v\,\mathrm{d}x = -\beta d_v \int_\Omega \nabla\left(1 - \frac{v^*}{v}\right)\cdot\nabla v\,\mathrm{d}x = -\beta d_v v^* \int_\Omega \frac{|\nabla v|^2}{v^2}\,\mathrm{d}x.$$

Next we consider the reaction terms. The equalities (23) together with setting $\alpha = br_v$ and $\beta = ar_u$ imply that

$$\begin{aligned}&\alpha r_u \int_\Omega (u-u^*)(1-u+av)\,\mathrm{d}x + \beta r_v \int_\Omega (v-v^*)(1-v-bu)\,\mathrm{d}x \\ &\quad = \alpha r_u \int_\Omega (u-u^*)(u^*-av^*-u+av)\,\mathrm{d}x + \beta r_v \int_\Omega (v-v^*)(v^*+bu^*-v-bu)\,\mathrm{d}x \\ &\quad = -\alpha r_u \int_\Omega (u-u^*)^2\,\mathrm{d}x - \beta r_v \int_\Omega (v-v^*)^2\,\mathrm{d}x,\end{aligned}$$

which in turn yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(u(t),v(t)) = & -\alpha d_u u^* \int_\Omega \varphi'(u)\frac{|\nabla u|^2}{u^2}\,\mathrm{d}x - \beta d_v v^* \int_\Omega \frac{|\nabla v|^2}{v^2}\,\mathrm{d}x \\ & -\alpha r_u \int_\Omega (u-u^*)^2\,\mathrm{d}x - \beta r_v \int_\Omega (v-v^*)^2\,\mathrm{d}x \le 0.\end{aligned} \tag{25}$$

We deduce from (25) that $V(u(t),v(t))$ is nonincreasing in time along the solution orbit $(u(t),v(t))$ and that $\partial_t V(u(t),v(t)) < 0$ for $(u,v) \neq (u^*,v^*)$.

Integrating the equation (25) on (t_*,T) for $T > t_*$ gives

$$\int_{t_*}^{T}\int_\Omega \left(\alpha r_u (u-u^*)^2 + \beta r_v (v-v^*)^2\right)\,\mathrm{d}x\,\mathrm{d}\tau \le V(u(t_*),v(t_*)).$$

Hence,

$$\int_{t_*}^{T}\int_\Omega \left((u-u^*)^2 + (v-v^*)^2\right)\,\mathrm{d}x\,\mathrm{d}\tau \le C_1$$

for each $T \ge t_*$ and $C_1 = V(u(t_*),v(t_*))/\min\{\alpha r_u, \beta r_v\} > 0$. By Lebesgue's monotone convergence theorem, we obtain

$$\int_{t_*}^{\infty}\int_\Omega \left((u-u^*)^2 + (v-v^*)^2\right)\,\mathrm{d}x\,\mathrm{d}\tau \le C_1. \tag{26}$$

Multiplying the equation for u by $u-u^*$ and integrating over Ω yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-u^*)^2\,\mathrm{d}x + d_u\int_\Omega \varphi'(u)|\nabla u|^2\,\mathrm{d}x + r_u\int_\Omega u(u-u^*)^2\,\mathrm{d}x & \\ = a r_u \int_\Omega u(u-u^*)(v-v^*)\,\mathrm{d}x & \quad . \\ \le a C_a r_u \int_\Omega |(u-u^*)(v-v^*)|\,\mathrm{d}x &\end{aligned}$$

Using Young's inequality we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-u^*)^2\,\mathrm{d}x \le a C_a r_u \int_\Omega \left((u-u^*)^2 + (v-v^*)^2\right)\,\mathrm{d}x.$$

Integrating this inequality over $(t,t+s)$ for $t \ge t_*$ and $s > 0$ gives

$$\int_\Omega (u-u^*)^2(t+s)\,\mathrm{d}x \le \int_\Omega (u-u^*)^2(t)\,\mathrm{d}x + C_2\int_t^{t+s}\int_\Omega \left((u-u^*)^2 + (v-v^*)^2\right)\,\mathrm{d}x\,\mathrm{d}\tau. \tag{27}$$

Similarly, we can show that

$$\int_\Omega (v-v^*)^2(t+s)\,\mathrm{d}x \le \int_\Omega (v-v^*)^2(t)\,\mathrm{d}x + C_2\int_t^{t+s}\int_\Omega \left((u-u^*)^2 + (v-v^*)^2\right)\,\mathrm{d}x\,\mathrm{d}\tau. \tag{28}$$

In both inequalities (27) and (28), $C_2 = \max\{a C_a r_u, b r_v\}$.

In view of (26), there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that $u(t_n) \to u^*$ and $v(t_n) \to v^*$ in $L^2(\Omega)$ as $t_n \to \infty$. Thus, for each $\varepsilon > 0$ there exist $T_1 \geq t_*$ such that for all $t_n \geq T_1$

$$\int_\Omega (u-u^*)^2(t_n)\,\mathrm{d}x \leq \frac{\varepsilon}{2} \quad \text{and} \quad \int_\Omega (v-v^*)^2(t_n)\,\mathrm{d}x \leq \frac{\varepsilon}{2}. \tag{29}$$

We also deduce from (26) that, for $\varepsilon > 0$ arbitrary, there exists $T_2 > 0$ such that for all $t_n \geq T_2$ and $s > 0$

$$\int_{t_n}^{t_n+s} \int_\Omega \left((u-u^*)^2 + (v-v^*)^2\right)\,\mathrm{d}x\,\mathrm{d}\tau \leq \frac{\varepsilon}{2C_2}. \tag{30}$$

In view of the estimates (29) and (30), we deduce from (27) and (28) that for each $t_n \geq \max\{T_1, T_2\}$ and for all $s > 0$

$$\int_\Omega (u-u^*)^2(t_n+s)\,\mathrm{d}x \leq \varepsilon \quad \text{and} \quad \int_\Omega (v-v^*)^2(t_n+s)\,\mathrm{d}x \leq \varepsilon. \tag{31}$$

Next we show that $u(t_n) \to u^*$ and $v(t_n) \to v^*$ in the L^2-norm along all subsequences $\{t_n\}$ such that $t_n \to \infty$. Indeed, let us suppose that there exists a subsequence $\{u(t_m)\}$ and a function u^{**} such that

$$u(t_m) \to u^{**} \quad \text{as} \quad t_m \to \infty$$

in $L^2(\Omega)$. Let

$$\varepsilon = \frac{1}{8}\|u^* - u^{**}\|^2_{L^2(\Omega)},$$

and let T_1 and T_2 be such that (31) is satisfied for all $t_n \geq \max\{T_1, T_2\}$ and $s > 0$. Moreover, let m be such that

$$\|u(t_m) - u^{**}\|^2_{L^2(\Omega)} \leq \varepsilon.$$

Choose $s = t_m - t_n$. Then, it follows from (31) that

$$\|u^* - u(t_m)\|^2_{L^2(\Omega)} \leq \varepsilon,$$

which implies that

$$\|u^* - u^{**}\|^2_{L^2(\Omega)} \leq 2(\|u^* - u(t_m)\|^2_{L^2(\Omega)} + \|u(t_m) - u^{**}\|^2_{L^2(\Omega)}) \leq 4\varepsilon = \frac{1}{2}\|u^* - u^{**}\|^2_{L^2(\Omega)},$$

and consequently that $u^{**} = u^*$. We deduce that $u(t) \to u^*$ and $v(t) \to v^*$ in $L^2(\Omega)$ as $t \to \infty$.

Moreover, the convergence $u(t) \to u^*$ and $v(t) \to v^*$ is uniform in $C(\overline{\Omega})$ as $t \to \infty$. Indeed, by the results of DiBenedetto [11] (Theorem 6.2 and the following corollary) we deduce that the sequence $\{u(t)\}_{t\geq 1}$ is equicontinuous in $\overline{\Omega}$ and thus relatively compact in $C(\overline{\Omega})$. Hence, we deduce the uniform convergence $u(t) \to u^*$ in $C(\overline{\Omega})$ as $t \to \infty$. The uniform convergence of $v(t)$ to v^* as $t \to \infty$ follows in a similar way. □

Lemma 3.4 (Rate of convergence for $0 < b < 1$)**.** *Let $0 < b < 1$, then the solution (u,v) of Problem ($\mathscr{P}$) converges exponentially fast to (u^*, v^*) in $[L^p(\Omega)]^2$ for all $p \geq 2$ as $t \to \infty$.*

Proof. Without loss of generality we assume that the solution (u,v) is different from (u^*,v^*). We deduce from the uniform convergence $u(t) \to u^*$ and $v(t) \to v^*$ in $C(\overline{\Omega})$ as $t \to \infty$ that

$$\text{for all } \mu \in (0, \min(u^*,v^*)), \text{ there exists } t_\mu > 0 \text{ such that } u(x,t),\, v(x,t) \geq \mu \text{ in } \overline{\Omega} \times [t_\mu, \infty). \quad (32)$$

Taylor's theorem implies that for each differentiable real valued strictly concave function f defined on an interval I

$$f(s_1) \leq f(s_2) + f'(s_2)(s_1 - s_2)$$

for all $s_1, s_2 \in I$, where the equality holds if and only if $s_1 = s_2$. Hence, taking $f(s) = -s\log s$ so that $f'(s) = -(\log s + 1)$ yields after some calculations

$$s_1 - s_2 \leq s_1(\log s_1 - \log s_2) \quad (33)$$

for each $s_1 \geq 0$ and $s_2 > 0$. Moreover, given a positive parameter p, the function $h : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$h(s,p) = \frac{\log s - \log p}{s - p} \quad (34)$$

is decreasing in s since $h_s(s,p) = (s - p - s\log(s/p))/(s(s-p)^2) \leq 0$ by (33). In view of (32), for a fixed $0 < \mu < \min\{u^*, v^*\}$ such that $u(x,t) \geq \mu$ and $v(x,t) \geq \mu$ in $\overline{\Omega} \times [t_\mu, \infty)$, it follows from the monotonicity of h that

$$h(u(x,t), u^*) \leq h(\mu, u^*) \quad \text{and} \quad h(v(x,t), v^*) \leq h(\mu, v^*) \quad (35)$$

for each $(x,t) \in \overline{\Omega} \times [t_\mu, \infty)$.

We have seen that the functional V defined by (24), i.e.

$$V(u,v) = \alpha \int_\Omega (u - u^* - u^* \log \frac{u}{u^*})\,\mathrm{d}x + \beta \int_\Omega (v - v^* - v^* \log \frac{v}{v^*})\,\mathrm{d}x \quad (36)$$

is a Lyapunov functional of the system. By using (33), (35) and (25) we obtain for each $t \geq t_\mu$ that

$$\begin{aligned} V(u(t),v(t)) &\leq \alpha \int_\Omega (u - u^*) \log \frac{u}{u^*}\,\mathrm{d}x + \beta \int_\Omega (v - v^*) \log \frac{v}{v^*}\,\mathrm{d}x \\ &= \alpha \int_\Omega (u - u^*)^2 h(u,u^*)\,\mathrm{d}x + \beta \int_\Omega (v - v^*)^2 h(v,v^*)\,\mathrm{d}x \\ &\leq \alpha h(\mu,u^*) \int_\Omega (u - u^*)^2\,\mathrm{d}x + \beta h(\mu,v^*) \int_\Omega (v - v^*)^2\,\mathrm{d}x \\ &\leq -K_1 \frac{\mathrm{d}}{\mathrm{d}t} V(u(t),v(t)) \end{aligned}$$

where $\mathrm{d}V/\mathrm{d}t$ is given in (25) and $K_1 = \max\{h(\mu,u^*)/r_u, h(\mu,v^*)/r_v\}$. Gronwall's inequality implies for $t \geq t_\mu$ that

$$V(u(t),v(t)) \leq V(u(t_\mu),v(t_\mu))e^{-(t-t_\mu)/K_1} = C_\mu e^{-(t-t_\mu)/K_1}, \quad (37)$$

where $C_\mu = V(u(t_\mu),v(t_\mu))$. We have now proved the exponential convergence to zero of the Lyapunov functional. The final step is to prove the exponential convergence of the solution to the steady state in the L^p-norm.

We recall Pinsker's inequality

$$p(s) = 2(1+2s)(s-1-\log s) - 3(s-1)^2 \geq 0, \tag{38}$$

which holds for all $s > 0$. Indeed, we can easily verify that $p(1) = p'(1) = 0$ and $p''(s) = 2(s-1)^2/s^2$ so that p is strictly convex on $(0,\infty)$ and attains its minimum at $s = 1$. We also refer to Exercise 2.3.26 on p. 58 in [6]. Since $u, v > 0$ in $\overline{\Omega} \times [t_\mu, \infty)$ as well as u and v are uniformly bounded from above (13), by substituting $s = u/u^*$ in Pinsker's inequality (38) and integrating in space we obtain that

$$\begin{aligned} 3\int_\Omega (u-u^*)^2\,\mathrm{d}x &\leq 2\int_\Omega (u^*+2u)(u-u^*-u^*\log\frac{u}{u^*})\,\mathrm{d}x \\ &\leq 2(u^*+2C_a)\int_\Omega (u-u^*-u^*\log\frac{u}{u^*})\,\mathrm{d}x, \end{aligned} \tag{39}$$

where the non-negativity of the last integral follows from (33) for $s_1 = u^*$ and $s_2 = u$. Similarly for $s = v/v^*$ we deduce that

$$3\int_\Omega (v-v^*)^2\,\mathrm{d}x \leq 2(v^*+2)\int_\Omega (v-v^*-v^*\log\frac{v}{v^*})\,\mathrm{d}x. \tag{40}$$

We deduce from (39) and (40) that

$$\begin{aligned} \alpha\int_\Omega (u-u^*-u^*\log\frac{u}{u^*})\,\mathrm{d}x + \beta\int_\Omega (v-v^*-v^*\log\frac{v}{v^*})\,\mathrm{d}x \geq \\ \frac{3\alpha}{2u^*+4C_a}\int_\Omega (u-u^*)^2\,\mathrm{d}x + \frac{3\beta}{2v^*+4}\int_\Omega (v-v^*)^2\,\mathrm{d}x \end{aligned}$$

so that by (36) we have for $t \geq t_\mu$

$$V(u(t),v(t)) \geq K_2\left(\|u-u^*\|^2_{L^2(\Omega)} + \|v-v^*\|^2_{L^2(\Omega)}\right) \tag{41}$$

where

$$K_2 = \min\left\{\frac{3\alpha}{2u^*+4C_a}, \frac{3\beta}{2v^*+4}\right\}.$$

From (37) and (41) we deduce the exponential convergence of the solution (u,v) to the positive steady state (u^*,v^*) in $L^2(\Omega)$, namely,

$$\|u-u^*\|^2_{L^2(\Omega)} + \|v-v^*\|^2_{L^2(\Omega)} \leq \frac{C_\mu}{K_2}e^{-(t-t_\mu)/K_1}.$$

The uniform bound (13), an interpolation inequality between L^2 and L^∞ (cf., Eq. 1.23 on p.13 in [21]), namely

$$\|u-u^*\|_{L^p(\Omega)} \leq \|u-u^*\|^{1-2/p}_{L^\infty(\Omega)}\|u-u^*\|^{2/p}_{L^2(\Omega)} \tag{42}$$

for $u-u^*$ and a similar inequality for $v-v^*$ yield

$$\|u-u^*\|_{L^p(\Omega)} + \|v-v^*\|_{L^p(\Omega)} \leq K_3 e^{-(t-t_\mu)/(pK_1)}$$

where

$$K_3 = \left((C_a+u^*)^{1-2/p} + (1+v^*)^{1-2/p}\right)\left(\frac{C_\mu}{K_2}\right)^{1/p}.$$

Hence, we conclude the exponential convergence of (u,v) to (u^*,v^*) in $L^p(\Omega)$ for all $p \geq 2$ as $t \to \infty$. □

3.3 Large time behaviour in the case that $b \geq 1$

Next, we show that whenever $b \geq 1$ the convergence of the solution of Problem $(\mathscr{P})$ to the stationary solution $(1,0)$ is uniform in $C(\overline{\Omega})$ as $t \to \infty$. Our approach is based upon a standard comparison principle. If $b \geq 1$, then similarly as in the proof of Lemma 3.4, a suitable Lyapunov functional can be used to explicitly compute the rate of convergence in $[L^p(\Omega)]^2$ for all $p \geq 2$. If $b > 1$, then the convergence is exponential, whereas if $b = 1$, the convergence is algebraic.

Lemma 3.5. *Let $b \geq 1$, then the solution (u,v) of Problem $(\mathscr{P})$ converges to $(1,0)$ in $[C(\overline{\Omega})]^2$ as $t \to \infty$.*

Proof. Let $\underline{u}$ be the nonnegative lower solution for the parabolic problem for u from the proof of Corollary 3.2, then $u \geq \underline{u}$ in $\Omega \times (0,\infty)$. We deduce from Lemma 3.1 that $\underline{u}(t) \to 1$ exponentially fast in $C(\overline{\Omega})$ as $t \to \infty$. Thus, for all $\vartheta \in (0,b)$ there exists $T_1 > 0$ such that for all $t \geq T_1$ we have

$$u \geq \underline{u} \geq 1 - \frac{\vartheta}{b} \tag{43}$$

in $\overline{\Omega} \times [T_1, \infty)$.

Next, let $\overline{v}$ be the solution of

$$\begin{cases} \overline{v}_t = d_v \Delta \overline{v} + r_v \overline{v}(\vartheta - \overline{v}) & \text{in } \Omega \times (T_1,\infty), \\ \dfrac{\partial \overline{v}}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (T_1,\infty), \\ \overline{v}(x,T_1) = v(x,T_1), \quad x \in \Omega. \end{cases}$$

We show that $\overline{v}$ is an upper solution for the parabolic problems for v. Indeed, since $b \geq 1$ and in view of (43), it follows that

$$\mathscr{L}_v(\overline{v}) = r_v \overline{v}(bu + \vartheta - 1) \geq r_v \overline{v}(b-1) \geq 0$$

in $\Omega \times [T_1,\infty)$, where $\mathscr{L}_v$ is defined in (20). Therefore, $\overline{v} \geq v$ in $\overline{\Omega} \times [T_1,\infty)$ by the comparison principle. Moreover, we deduce from Lemma 3.1 that $\overline{v} \to \vartheta$ exponentially fast in $C(\overline{\Omega})$ as $t \to \infty$. It follows from this uniform convergence that there exists $T_2 \geq T_1$ such that for all $t \geq T_2$

$$0 \leq v \leq \overline{v} \leq 2\vartheta \tag{44}$$

in $\overline{\Omega} \times [T_2,\infty)$. Since (44) holds for all $\vartheta > 0$, we deduce that

$$v(t) \to 0 \quad \text{uniformly in } C(\overline{\Omega}) \text{ as } t \to \infty. \tag{45}$$

Finally, we find an upper solution $\overline{u}$ for the parabolic problem for u. We note that, it follows from the uniform convergence (45) that for $\vartheta > 0$ there exists $T_3 > 0$ such that for all $t \geq T_3$ we have

$$0 \leq v \leq \frac{\vartheta}{a} \tag{46}$$

in $\overline{\Omega} \times [T_3,\infty)$. Let $\overline{u}$ be the solution to

$$\begin{cases} \overline{u}_t = d_u \Delta \varphi(\overline{u}) + r_u \overline{u}(1 + \vartheta - \overline{u}) & \text{in } \Omega \times (T_3,\infty), \\ \dfrac{\partial \varphi(\overline{u})}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (T_3,\infty), \\ \overline{u}(x,T_3) = u(x,T_3), \quad x \in \Omega. \end{cases}$$

In view of (46) we deduce that

$$\mathscr{L}_u(\overline{u}) = r_u\overline{u}(\vartheta - av) \geq 0$$

in $\Omega\times[T_3,\infty]$, where $\mathscr{L}_u$ is defined in (19). Thus, $\overline{u}\geq u$ in $\overline{\Omega}\times[T_3,\infty)$ by the comparison principle. Moreover, it follows from Lemma 3.1 that $\overline{u}\to 1+\vartheta$ exponentially fast in $C(\overline{\Omega})$ as $t\to\infty$. Thus, there exists $T_4\geq T_3$ such that for all $t\geq T_4$

$$u \leq \overline{u} \leq 1+2\vartheta \tag{47}$$

in $\overline{\Omega}\times[T_4,\infty)$. The estimates (43) and (47) imply the existence of $T_5=\max\{T_1,T_4\}$ such that for all $t\geq T_5$ we have

$$1-\frac{\vartheta}{b} \leq u \leq 1+2\vartheta$$

in $\overline{\Omega}\times[T_5,\infty)$. Since $\vartheta>0$ can be chosen arbitrarily, we deduce that

$$u(t)\to 1 \quad \text{uniformly in } C(\overline{\Omega}) \text{ as } t\to\infty. \tag{48}$$

Thus, $(u,v)(t)\to(1,0)$ uniformly in $C(\overline{\Omega})$ as $t\to\infty$. □

Lemma 3.6 (Rate of convergence for $b>1$)**.** *Let $b>1$, then the solution (u,v) of Problem $(\mathscr{P})$ converges exponentially fast to $(1,0)$ in $[L^p(\Omega)]^2$ for all $p\geq 2$ as $t\to\infty$.*

Proof. We will follow the proof of Lemma 3.4. Without loss of generality we assume that the solution (u,v) is different from $(1,0)$. It follows from the uniform convergence (48) that

for all $\mu<1$, there exists $t_\mu>0$ such that $u(x,t)\geq\mu$ in $\overline{\Omega}\times[t_\mu,\infty)$.

We consider the functional

$$V(u,v) = \alpha\int_\Omega (u-1-\log u)\,\mathrm{d}x + \beta\int_\Omega v\,\mathrm{d}x, \tag{49}$$

on the time interval $[t_\mu,\infty)$. We obtain that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(u(t),v(t)) &= \alpha\int_\Omega \frac{u-1}{u}u_t\,\mathrm{d}x + \beta\int_\Omega v_t\,\mathrm{d}x\\
&= \alpha\int_\Omega \frac{u-1}{u}\left(d_u\Delta\varphi(u) + r_u u(1-u+av)\right)\mathrm{d}x + \beta\int_\Omega \left(d_v\Delta v + r_v v(1-v-bu)\right)\mathrm{d}x\\
&= -\alpha d_u\int_\Omega \varphi'(u)\frac{|\nabla u|^2}{u^2} - \alpha r_u\int_\Omega (u-1)^2\,\mathrm{d}x - \beta r_v\int_\Omega v^2\,\mathrm{d}x - \alpha\beta\,(1-1/b)\int_\Omega v\,\mathrm{d}x\\
&\leq 0
\end{aligned} \tag{50}$$

for $\alpha=br_v$, $\beta=ar_u$, $1/b\leq 1$ and $t\geq t_\mu$. Moreover, in the case that $b>1$ and for each $t\geq t_\mu$ we can use the inequality $s-1\leq s\log s$ for $s\geq 0$, which is obtained from (33) for $s_1=s$ and $s_2=1$, to calculate

$$\begin{aligned}
V(u(t),v(t)) &\leq \alpha\int_\Omega (u-1)\log u\,\mathrm{d}x + \beta\int_\Omega v\,\mathrm{d}x\\
&= \alpha\int_\Omega (u-1)^2\frac{\log u}{u-1}\,\mathrm{d}x + \beta\int_\Omega v\,\mathrm{d}x\\
&\leq \alpha h(\mu,1)\int_\Omega (u-1)^2\,\mathrm{d}x + \beta\int_\Omega v\,\mathrm{d}x\\
&\leq -K_4\frac{\mathrm{d}}{\mathrm{d}t}V(u(t),v(t)),
\end{aligned} \tag{51}$$

where the function h is defined by (34) and $K_4 = \max\{h(\mu,1)/r_u, 1/(\alpha(1-1/b))\}$. Gronwall's inequality implies for $t \geq t_\mu$ that

$$V(u(t),v(t)) \leq V(u(t_\mu),v(t_\mu))e^{-(t-t_\mu)/K_4} = C_\mu e^{-(t-t_\mu)/K_4}, \tag{52}$$

where $C_\mu = V(u(t_\mu),v(t_\mu))$. For $t \geq t_\mu$, Pinsker's inequality (38) yields

$$\frac{3\alpha}{2(1+2C_a)}\int_\Omega (u-1)^2\,\mathrm{d}x \leq \alpha\int_\Omega (u-1-\log u)\,\mathrm{d}x. \tag{53}$$

We deduce from (49), (52), (53) and the trivial observation $v^2 \leq v$ for $0 \leq v \leq 1$ that

$$\frac{3\alpha}{2(1+2C_a)}\|u-1\|^2_{L^2(\Omega)} + \beta\|v\|^2_{L^2(\Omega)} \leq C_\mu e^{-(t-t_\mu)/K_4},$$

i.e.,

$$\|u-1\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \leq \frac{C_\mu}{K_5}e^{-(t-t_\mu)/K_4},$$

where

$$K_5 = \min\left\{\frac{3\alpha}{2+4C_a}, \beta\right\}. \tag{54}$$

Finally, we deduce from the interpolation inequality (42) and the uniform bound (13) that

$$\|u-1\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \leq K_6 e^{-(t-t_\mu)/(pK_4)}$$

where

$$K_6 = \left((C_a+1)^{1-2/p}+1\right)\left(\frac{C_\mu}{K_5}\right)^{1/p}.$$

Hence, we conclude the exponential convergence of (u,v) to $(1,0)$ in $L^p(\Omega)$ for all $p \geq 1$ as $t \to \infty$. □

The case when $b = 1$ is more delicate and the procedure above does not work. In this case, the good term $(1-1/b)\int_\Omega v$ in the time derivative of $V(u,v)$ in (50) vanishes. Nevertheless, we can still show the existence of a constant $C > 0$ such that

$$V^2(t) \leq CD(t)$$

for large enough time t, where $V(t) = V(u(t),v(t))$ and $D(t) = D(u(t),v(t)) = -\mathrm{d}V(u(t),v(t))/\mathrm{d}t$. We obtain that $V(t) \approx C/t$, in other words, we deduce the algebraic convergence to zero of the Lyapunov functional as $t \to \infty$.

Lemma 3.7 (Rate of convergence for $b = 1$)**.** *Let $b = 1$, then the solution (u,v) of Problem $(\mathscr{P})$ converges algebraically fast to $(1,0)$ in $[L^p(\Omega)]^2$ for all $p \geq 2$ as $t \to \infty$.*

Proof. As in the proof of Lemma 3.6, the uniform convergence of the solution (u,v) of Problem $\mathscr{P}$ to the equilibrium $(1,0)$ implies that

for all $\mu < 1$, there exists $t_\mu > 0$ such that $u(x,t) \geq \mu$ in $\overline{\Omega} \times [t_\mu, \infty)$.

For the functional $V(u,v)$ defined by (49) we can repeat the same calculations as in (50) and (51). In particular, we obtain

$$D(u(t),v(t)) = -\frac{\mathrm{d}}{\mathrm{d}t}V(u(t),v(t)) = \alpha d_u \int_\Omega \varphi'(u)\frac{|\nabla u|^2}{u^2} + \alpha r_u \int_\Omega (u-1)^2\,\mathrm{d}x + \beta r_v \int_\Omega v^2\,\mathrm{d}x$$

and

$$V(u(t),v(t)) \le \alpha h(\mu,1)\int_\Omega (u-1)^2\,\mathrm{d}x + \beta\int_\Omega v\,\mathrm{d}x, \tag{55}$$

where $0<\mu<1$ and $t\ge t_\mu$. We deduce from the inequality

$$\int_\Omega v^2(x)\,\mathrm{d}x \ge \frac{1}{|\Omega|}\left(\int_\Omega v(x)\,\mathrm{d}x\right)^2$$

that

$$D(u(t),v(t)) \ge \alpha r_u \int_\Omega (u-1)^2\,\mathrm{d}x + \frac{\beta r_v}{|\Omega|}\left(\int_\Omega v\,\mathrm{d}x\right)^2. \tag{56}$$

In the sequel, we will use the notations $V=V(u(t),v(t))$, $D=D(u(t),v(t))$, $V_1=\int_\Omega (u-1)^2\,\mathrm{d}x$ and $V_2=\int_\Omega v\,\mathrm{d}x$. Hence, (55) and (56) can be written shortly as

$$V \le \alpha h(\mu,1)V_1 + \beta V_2 \tag{57}$$

and

$$D \ge \alpha r_u V_1 + \frac{\beta r_v}{|\Omega|}V_2^2. \tag{58}$$

Next, we distinguish two cases.

Case I. Suppose that $V_1\ge 1$ or $V_2\ge 1$. Then, $D\ge K_1=\min\{\alpha r_u,\beta r_v/|\Omega|\}$. Moreover, the uniform estimate (13) implies that $0\le V_1\le (C_a^2+1)|\Omega|$ and $0\le V_2\le|\Omega|$, where $C_a=1+a$. We deduce from (57) that $V\le K_2$ where K_2 depends on α, $h(\mu,1)$, C_a, β and $|\Omega|$. Hence, we obtain

$$V^2 \le K_2^2 = C_1K_1 \le C_1 D, \quad \text{for } C_1 = K_2^2/K_1. \tag{59}$$

Case II. Suppose that $0\le V_1<1$ and $0\le V_2<1$. Then, $V_1\ge V_1^2$ and by using the trivial inequality $x^2+y^2\ge (x+y)^2/2$ we estimate D in (58) from below by

$$D \ge \alpha r_u V_1^2 + \beta r_v V_2^2 \ge K_3(V_1+V_2)^2,$$

where $K_3=\min\{\alpha r_u,\beta r_v/|\Omega|\}/2$. On the other hand, in view of (57),

$$V \le K_4(V_1+V_2),$$

where $K_4=\max\{\alpha h(\mu,1),\beta\}$. These two estimates imply

$$V^2 \le K_4^2(V_1+V_2)^2 = C_2K_3(V_1+V_2)^2 \le C_2 D, \quad \text{for } C_2 = K_4^2/K_3. \tag{60}$$

In view of the estimates (59) and (60), we can take $C=\max\{C_1,C_2\}$ so that $V^2\le CD$. Hence, we proved that for $t\ge t_\mu$

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le -\frac{1}{C}V^2(t).$$

We deduce that

$$V(t) \leq \frac{C}{t - t_\mu + C_3} \leq \frac{C}{t - t_\mu}$$

for $t \geq t_\mu$, where $C_3 > 0$ depends on the constant C and on $u(t_\mu)$.

Similarly as in the proof of Lemma 3.6, we can use Pinsker's inequality (38) to derive the lower bound for $V(u,v)$ and so to deduce

$$\|u-1\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \leq \frac{C}{K_5} \frac{1}{t - t_\mu},$$

as well as

$$\|u-1\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \leq \frac{K_6}{t - t_\mu}$$

where

$$K_6 = \left((C_a + 1)^{1-2/p} + 1 \right) \left(\frac{C}{K_5} \right)^{1/p}$$

and K_5 is given by (54). □

3.4 Convergence to equilibrium as $t \to \infty$ under the assumption $(\widetilde{H}_\varphi)$

Let us assume the hypothesis $(\widetilde{H}_\varphi)$ instead of (H_φ). Then, Problem $(\mathscr{P})$ turns out to be uniformly parabolic and possesses a unique classical solution $(u,v) \in [C^{2,1}(\overline{Q}_T)]^2$, [19] (Chap. V, Theorem 7.4). In the case when $\varphi(u) = u$, we refer the readers to [17] for additional details. Moreover, by the strong maximum principle [19] the solution $(u(\cdot,t), v(\cdot,t))$ is positive in Ω for all $t > 0$ so that we do not have to show eventual positivity of the solution as we did in Section 3.1. Hence, the functionals $V(u,v)$ given by (4) and (5) are well defined for all times $t > 0$. Let us, moreover, assume that $u_0 > 0$ and $v_0 > 0$ in Ω. Then, $V(u,v)$ is defined for all $t \geq 0$.

By repeating the proofs of Lemmas 3.3 and 3.5, we can show that

$$(u(t), v(t)) \to \begin{cases} (u^*, v^*) & \text{if } 0 < b < 1, \\ (1,0) & \text{if } b \geq 1, \end{cases}$$

uniformly in $[C(\overline{\Omega})]^2$ as $t \to \infty$, where $(u^*, v^*) = ((1+a)/(1+ab), (1-b)/(1+ab))$.

Similar calculations as those in Lemmas 3.4 and 3.6 can be used in order to show the exponential convergence of the solution (u,v) to its respective steady state solution in both cases $0 < b < 1$ and $b > 1$. Indeed, for a chosen $\mu < \min\{u^*, v^*\}$ in the case when $0 < b < 1$ and $\mu < 1$ in the case when $b > 1$ we can find $t_\mu > 0$ so that we can apply (35) to derive (37), resp. (52). However, $V(u,v)$ is defined for all times $t \geq 0$ and it is nonincreasing in time. Thus, if $0 < b < 1$, then (37) can be further estimated from above,

$$\begin{aligned} V(u(t), v(t)) &\leq V(u(t_\mu), v(t_\mu)) e^{-(t-t_\mu)/K_1} \\ &\leq V(u_0, v_0) e^{-(t-t_\mu)/K_1}. \end{aligned}$$

Similarly, if $b > 1$, instead of (52) we obtain

$$\begin{aligned} V(u(t), v(t)) &\leq V(u(t_\mu), v(t_\mu)) e^{-(t-t_\mu)/K_4} \\ &\leq V(u_0, v_0) e^{-(t-t_\mu)/K_4}. \end{aligned}$$

However, we cannot remove the constant t_μ from the estimates. A lower bound for $V(u,v)$ in terms of the L^p-distance of the solution (u,v) from the steady state solution is the same as in the degenerate parabolic case, see Lemmas 3.4 and 3.6.

4 Concluding remarks

We have studied the well-posedness of the degenerate, nonlinear diffusion system $(\mathscr{P})$ for the Neolithic evolution of the farming and hunting-gathering populations u and v, respectively. Furthermore, we studied the asymptotic behaviour of the solution (u,v) of Problem $(\mathscr{P})$. We found that (u,v) always converges to a spatially homogeneous steady state as $t \to \infty$. In particular, if $b \geq 1$ and $\varphi(u) = p(u)u$ where $p = p(u)$ is the probability density function satisfying (i)-(iii), then

$$\lim_{t\to\infty}(u,v)(x,t) = (1,0)$$

uniformly in Ω. In view of (2), this scenario corresponds to the case when $g \geq r_H$ and (F,H) converges to the state $(1,0)$ as $t \to \infty$.

If $p(F)$ in the system (2) is specified as $p(F) = F/(F+F_c)$, where a positive constant F_c is assumed to represent a level of development of farming and food-producing technology, see [15, 18] for additional details, then the convergence result implies that the asymptotic behaviour of the solution (F,H) of the system (2) is independent of F_c. However, we remark that the transient behaviour of (F,H) depends on the values of parameters. Indeed, let us suppose that Ω is rather large. Then, if the parameters of the system (2) are suitably chosen and F_c is relatively small, the spatial shape of the solution (F,H) becomes radially symmetric, see Fig. 1. On the other hand, for the same parameters and F_c relatively large we can observe breaking (instability) of radial symmetry in (F,H), see Fig. 2. It can be numerically confirmed that this phenomena never occurs in (1), namely, in the case of linear diffusion in the equation for farmers F in (2). We emphasise that this is a striking difference between the two systems (1) and (2). We therefore call such an instability "a nonlinear diffusion-induced instability". We propose to try to understand why nonlinear diffusion generates such an instability in future work.

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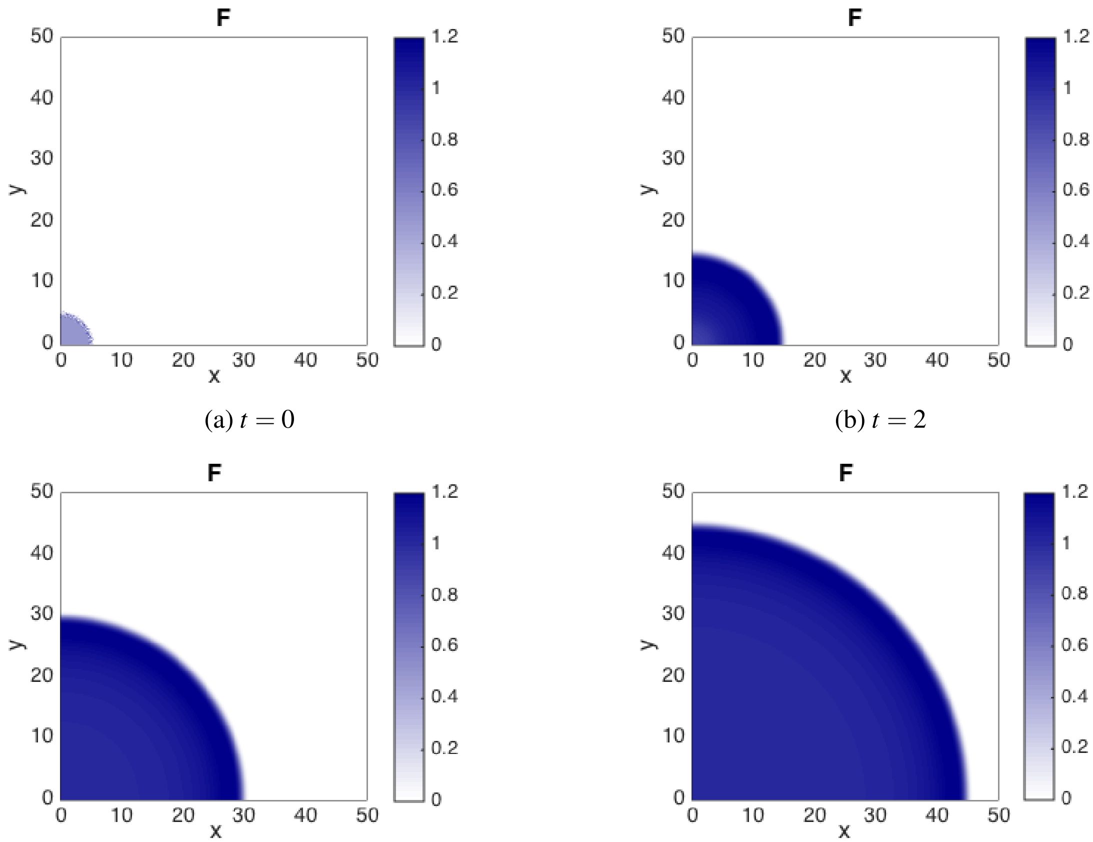

(a) $t = 0$ (b) $t = 2$

(c) $t = 5$ (d) $t = 8$

Fig. 1: Spatial pattern of F of the system (2) with $p(F) = F/(F + F_c)$ at different times where $d_F = 0.1$, $d_H = 1.0$, $r_F = r_H = 1.0$, $s = 25$, $g = 22$ and $F_c = 1.0$.

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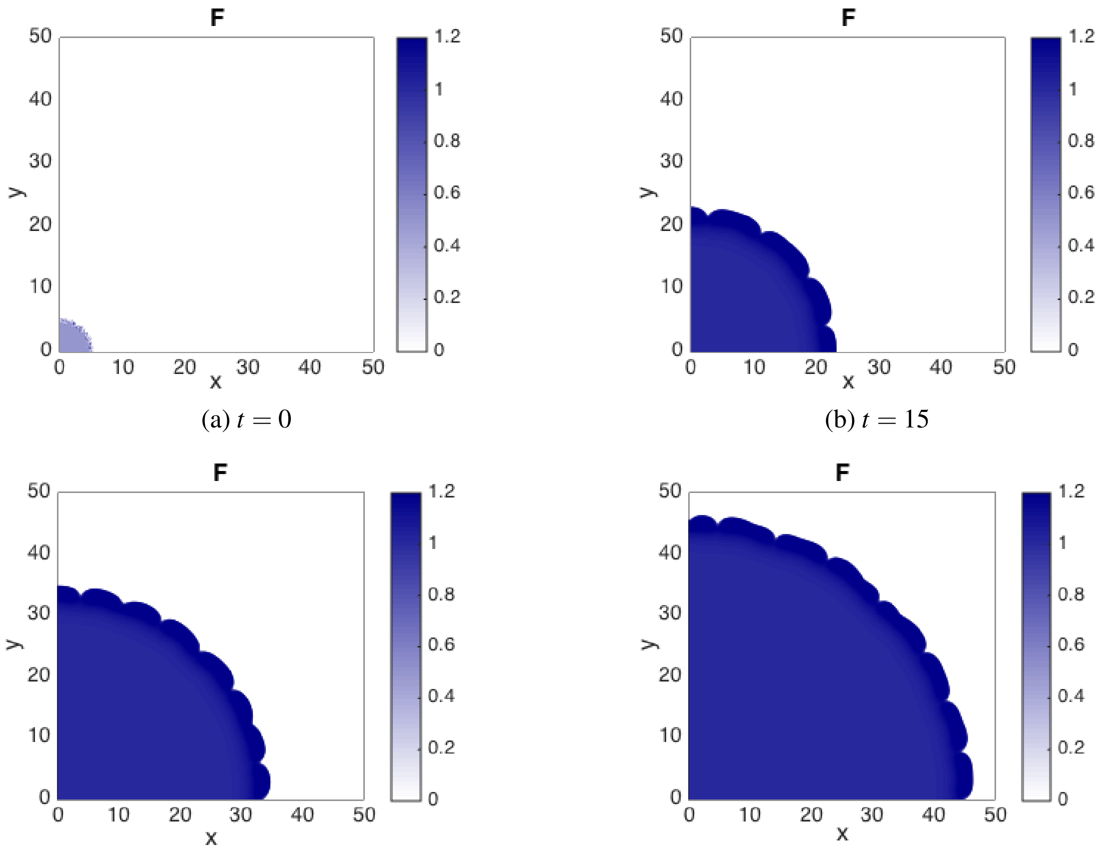

(a) $t = 0$ (b) $t = 15$

(c) $t = 25$ (d) $t = 35$

Fig. 2: Spatial pattern of F of the system (2) with $p(F) = F/(F + F_c)$ at different times where $d_F = 0.1$, $d_H = 1.0$, $r_F = r_H = 1.0$, $s = 25$, $g = 22$ and $F_c = 5.0$.

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